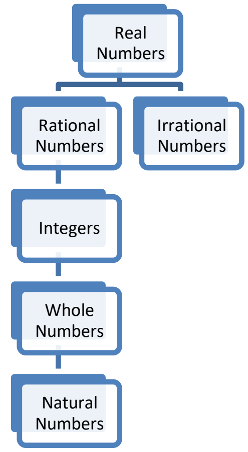
1.0 – Classifying Real Numbers and Rational Exponents Review

Classifying Real Numbers:



Natural Numbers – The counting numbers

1, 2, 3, 4, ...

Whole Numbers – Zero and the counting numbers

0, 1, 2, 3, 4, ...

Integers – Neg counting numbers, 0, pos counting numbers

.... −4, −3, −2, −1, 0, 1, 2, 3, 4 .....

Rational Numbers – All numbers that can be written as a fraction

(*if written as a decimal can it be converted to a fraction?) $\frac{3}{4}$

→ decimals that end (terminate) $0.2 = \frac{1}{5}$

→ decimals that repeat $0.\overline{3} = \frac{1}{3}$

Irrational Numbers – Everything else that can't be written as a fraction

eg: $\sqrt{2}$, $\sqrt[3]{5}$, π, 0.12112111 2... (pattern but doesn't repeat)

Real Numbers – All numbers

Rational and irrational numbers

Example 1 – Consider the list of numbers: -2, 0, 1, $\frac{4}{5}$, 0.777, -2.5, $\sqrt{17}$, $\sqrt[3]{8}$, 4.14562972....

4.1231... 2

List all:

a) Natural Numbers

1, $\sqrt[3]{8}$

b) Whole Numbers

0, 1, $\sqrt[3]{8}$

c) Integers

−2, 0, 1, $\sqrt[3]{8}$

d) Rational numbers

−2, 0, 1 $\sqrt[3]{8}$, $\frac{4}{5}$, 0.777, −2.5

e) Irrational Numbers

$\sqrt{17}$, 4.14562972....

f) Real numbers

all of them

Example 2 – State whether each statement is **true or false**.

a) Every integer is a natural number — False

neg count, 0, counting — counting

b) All whole numbers are integers — True

0, pos — neg, 0, pos

c) Every real number is a rational number — False

all numbers — can be written as fractions

Rational Exponents:

When n is a natural number and x is a rational number, $x^{\frac{1}{n}} = \sqrt[n]{x}$

Example 3 – Write each power in radical form and evaluate without using a calculator:

a) $1000^{\frac{1}{3}}$

$\sqrt[3]{1000}$

$= 10$

b) $25^{0.5}$

$25^{\frac{1}{2}}$

$\sqrt{25}$

$= 5$

c) $(-8)^{\frac{1}{3}}$

$\sqrt[3]{-8}$

$= -2$

d) $\left(\frac{16}{81}\right)^{\frac{1}{4}}$

$\frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$

When m and n are natural numbers, and x is a rational number, $x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$

Example 4 – Write the following in radical form:

flower power

a) $26^{\frac{2}{5}}$ exp root

$\left(\sqrt[5]{26}\right)^2$

b) $25^{\frac{3}{2}}$

$\left(\sqrt{25}\right)^3 = 5^3 = 125$

Example 5 – Write the following in exponent form:

a) $\left(\sqrt[2]{6}\right)^5 = 6^{\frac{5}{2}}$

b) $\left(\sqrt[4]{19}\right)^3 = 19^{\frac{3}{4}}$

To evaluate a power with a negative rational exponent,

1) Write with a positive exponent (recall $x^{-1} = \frac{1}{x}$ $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$)
2) Re-write into radical form
3) Work from the inside out
4) Write answer with no exponents

Example 6 – Simplify

a) $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$ Flip to remove neg

$= \left(\frac{16}{9}\right)^{\frac{3}{2}}$

$= \frac{16^{\frac{3}{2}}}{9^{\frac{3}{2}}}$

$= \frac{\left(\sqrt{16}\right)^3}{\left(\sqrt{9}\right)^3}$

$= \frac{4^3}{3^3}$

$= \boxed{\frac{64}{27}}$

b) $\frac{16^{-\frac{5}{4}}}{1}$

$= \left(\frac{1}{16}\right)^{\frac{5}{4}}$

$= \frac{1^{\frac{5}{4}}}{16^{\frac{5}{4}}}$

$= \frac{\left(\sqrt[4]{1}\right)^5}{\left(\sqrt[4]{16}\right)^5}$

$= \frac{1^5}{2^5}$

$= \boxed{\frac{1}{32}}$

c) $-25^{-1.5}$ $1.5 = \frac{3}{2}$

$= -\left(\frac{25}{1}\right)^{-\frac{3}{2}}$

$= -\left(\frac{1}{25}\right)^{\frac{3}{2}}$

$= -\frac{1}{\left(\sqrt{25}\right)^3}$

$= -\frac{1}{5^3}$

$= \boxed{-\frac{1}{125}}$

1.4A – Simplifying Radicals Review and Preview

Example 1 – Identify and define all parts of the radical, then simplify:

$5\sqrt[3]{8}$

index, radicand, coefficient, radical sign

$5\sqrt[3]{8} = 5 \cdot 2 = 10$

Radical Properties from Math 10:

1) $a^{\frac{1}{n}} = \sqrt[n]{a}$ as discussed in previous notes

2) $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m$ Example: $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = (2)^3 = 8$

3) $a^{-\frac{m}{n}} = (a^{\frac{1}{n}})^{-m} = (\sqrt[n]{a})^{-m} = \frac{1}{(\sqrt[n]{a})^m}$ Example: $27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \frac{1}{9}$

4) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ Example: $\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}$

5) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ Example: $\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

12 → 2, 6; 6 → 2, 3

In this chapter, it is helpful to know the following:

Perfect squares up to 144:

$1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, 11^2, 12^2$

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144.... $x^2, x^4, x^6, x^8 \ldots\ldots x^{\text{even}}$

Perfect cubes up to 125:

$1^3, 2^3, 3^3, 4^3, 5^3$

1, 8, 27, 64, 125.... $x^3, x^6, x^9, x^{12} \ldots\ldots x^{\text{divide by 3}}$

Perfect powers of 4 up to 81:

$1^4, 2^4, 3^4$

1, 16, 81 .... $x^4, x^8, x^{12}, x^{16} \ldots\ldots x^{\text{divide by 4}}$

When a radical is simplified, an *entire radical* is changed to a *mixed radical* (or a mixed radical is further simplified).

1) Find the largest perfect **square** factor of the radicand ("under the root")
2) Rewrite the radicand as a product of perfect **square** x other factor
3) **Square** root the perfect **square** and write that new answer out front of the root
4) If there is already a number out front of the root, multiply them together
   *(Similar process for **cube** roots, etc.)*

Example 2 – Simplify:

a) $\sqrt{50}$

$= \sqrt[2]{25 \cdot 2}$

$= \sqrt{25} \cdot \sqrt{2}$

$= 5\sqrt{2}$

or 50 → 2, 25; 25 → 5, 5

index is 2, so looking for matching pairs, can remove that number from radical sign

b) $\sqrt{20}$

$= \sqrt{4 \cdot 5}$

$= \sqrt{4} \times \sqrt{5}$

$= 2\sqrt{5}$

or 20 → 2, 10; 10 → 2, 5 → $2\sqrt{5}$

c) $\frac{5}{6}\sqrt{18}$

18 → 2, 9; 9 → 3, 3

$\frac{5}{6}\sqrt{9 \cdot 2}$

$\frac{5}{6}\sqrt{9} \times \sqrt{2}$

$\frac{5}{6} \times \frac{3}{1}\sqrt{2}$

or $\frac{5}{\cancel{6}_2} \times \frac{\cancel{3}^1}{1} = \frac{5}{2}$

$= \frac{15 \div 3}{6 \div 3}\sqrt{2}$

$= \frac{5}{2}\sqrt{2}$ or $\frac{5\sqrt{2}}{2}$

why is this allowed? we can "odd" root negatives! $(-3)^3 = -3 \times -3 \times -3 = -27 \therefore \sqrt[3]{-27} = -3$

Example 3 – Simplify:

a) $\sqrt[3]{40}$

$= \sqrt[3]{8 \cdot 5}$

$= 2\sqrt[3]{5}$

b) $2\sqrt[3]{-54}$

$= 2\sqrt[3]{-27 \cdot 2}$

$= 2 \times (-3)\sqrt[3]{2}$

$-6\sqrt[3]{2}$

c) $4\sqrt[4]{162}$

$4 \times \sqrt[4]{81 \cdot 2}$

$4 \cdot 3 \sqrt[4]{2}$

$12\sqrt[4]{2}$

or 162 = 2 · 81, 81 = 3 · 27, 27 = 3 · 9, 9 = 3 · 3

What if there are variables in the expression?

Roots of Positive Powers of x – Case 1: When $x \geq 0$ in $\sqrt{x^n}$ with n a positive integer.
The square roots of negative numbers are undefined in the set of real numbers. Therefore, if $x \geq 0$, simplification is easier to realize. We will work only with "Case 1" for most of the chapter. You will know it's case 1 because the question will say "assume all variables represent positive numbers" or "all variables represent non-negative real numbers."

For example:

$\sqrt{x} = \sqrt{x}$

$\sqrt{x^2} = x$

$\sqrt{x^3} = \sqrt{x^2 x} = x\sqrt{x}$

$\sqrt{x^4} = \sqrt{(x^2)^2} = x^2$

$\sqrt{x^5} = \sqrt{x^4 \cdot x} = x^2\sqrt{x}$

$\sqrt{x^6} = \sqrt{(x^3)^2} = x^3$

Short cut $= \sqrt{x^6} = x^{\frac{6}{2}} = x^3$

↳ divide power by index, if odd one $\sqrt{x}$ left over

$\sqrt[3]{x} = \sqrt[3]{x}$

$\sqrt[3]{x^2} = \sqrt[3]{x^2}$

$\sqrt[3]{x^3} = x$

$\sqrt[3]{x^4} = \sqrt[3]{x^3 \cdot x} = x\sqrt[3]{x}$

$\sqrt[3]{x^5} = \sqrt[3]{x^3 \cdot x^2} = x\sqrt[3]{x^2}$

$\sqrt[3]{x^6} = \sqrt[3]{(x^2)^3} = x^2$

Short cut $\sqrt[3]{x^6} = x^{\frac{6}{3}} = x^2$

ie divide by 3, if not divisible remove $\sqrt{x}$ or $\sqrt{x^2}$ so its divisible.

Divide by power of index

A note about Case 2: When x is any real number in $\sqrt{x^n}$, with n a positive integer
This case introduces complications and restrictions which we will avoid by working with variables that represent non-negative values. See section 1.3 in your workbook for an explanation of Case 2.

Example 4 – Simplify. Assume all variables represent positive numbers.

a) $\sqrt{x^7y^3}$

$\sqrt{x^6 x\, y^2 y}$

$x^3 y\sqrt{xy}$

b) $\sqrt[3]{x^{14}y^9}$

$= \sqrt[3]{x^{12}x^2 y^9}$

$= x^4 y^3 \sqrt[3]{x^2}$

$x^{14} = x^{12} \cdot x^2$

largest power that ÷ 3

"left overs" (ie: 12+2 = 14)

c) $\sqrt[4]{x^{11}}$

$= \sqrt[4]{x^8 y^3}$

$= x^2\sqrt[4]{y^3}$

1.4B – Simplifying Radicals

Example 1 – Simplify

can we do this? Yes since exp odd.

a) $\sqrt{8}$

$= \sqrt{4 \cdot 2}$

$= 2\sqrt{2}$

b) $\sqrt{27}$

$= \sqrt{9 \cdot 3}$

$= 3\sqrt{3}$

c) $3\sqrt{52}$

$= 3\sqrt{4 \cdot 13}$

$= 3 \times 2\sqrt{13}$

$= 6\sqrt{13}$

d) $\sqrt[3]{24}$

$= \sqrt[3]{8 \cdot 3}$

$= 2\sqrt[3]{3}$

e) $5\sqrt[3]{-81}$

$= 5\sqrt[3]{-27 \cdot 3}$

$= 5(-3)\sqrt[3]{3}$

$= -15\sqrt[3]{3}$

f) $\sqrt[4]{32}$

$= \sqrt[4]{16 \cdot 2}$

$= 2\sqrt[4]{2}$

Example 2 – Simplify (assume variables are positive)

a) $\sqrt{18x^3y^6}$

$= \sqrt{9 \cdot 2\, x^2 x\, y^6}$

$= 3xy^3\sqrt{2x}$

b) $\sqrt{63n^7p^4}$

$= \sqrt{9 \cdot 7\, n^6 n\, p^4}$

$= 3n^3p^2\sqrt{7n}$

c) $\sqrt{32x^8y^{11}}$

$= \sqrt{16 \cdot 2\, x^8 y^{10} y^1}$

$= 4x^4y^5\sqrt{2y}$

Check index each time

d) $\sqrt[3]{40a^4b^8c^{15}}$

$= \sqrt[3]{8 \cdot 5\, a^3 a\, b^6 b^2\, c^{15}}$

$= 2ab^2c^5\sqrt[3]{5ab^2}$

e) $\sqrt[3]{54a^5b^{10}}$

$= \sqrt[3]{27 \cdot 2 \cdot a^3a^2\, b^9 b^1}$

$= 3ab^3\sqrt[3]{2a^2b}$

f) $\sqrt[3]{\dfrac{x^{13}}{64}}$

$= \dfrac{\sqrt[3]{x^{12}x^1}}{\sqrt[3]{64}}$

$= \dfrac{x^4\sqrt[3]{x}}{4}$

g) $\sqrt[4]{162x^3y^{11}z^5}$

$= \sqrt[4]{81 \cdot 2\, x^3\, y^8 y^3\, z^4 z^1}$

$= 3y^2z\sqrt[4]{2x^3y^3z}$

h) $\sqrt[4]{m^7}$

$= \sqrt[4]{m^4m^3}$

$= m\sqrt[4]{m^3}$

Changing Mixed Radicals to Entire Radicals

Usually we work with radicals in simplest form (mixed radicals, with the smallest possible radicand). But we can change mixed radicals to entire radicals. This allows us to compare radicals.

Example 3 – Without using a calculator, determine which number is larger. (Change to entire radicals and compare)

$4\sqrt{3}$ or $3\sqrt{5}$

$= \sqrt{4^2 \times 3}$ $\qquad = \sqrt{3^2 \times 5}$

$= \sqrt{16 \times 3}$ $\qquad = \sqrt{9 \times 5}$

$= \sqrt{48}$ $\quad > \quad$ $= \sqrt{45}$

$4\sqrt{3}$ is larger

Example 4 – Change to Entire (assume variables are positive)

add index on as exponent when moving a number under the radical

must be a neg at the end keep - outside

a) $5\sqrt[2]{2}$

$= \sqrt{5^2 \times 2}$

$= \sqrt{25 \times 2}$

$= \sqrt{50}$

b) $7\sqrt{3}$

$= \sqrt{7^2 \times 3}$

$= \sqrt{49 \times 3}$

$= \sqrt{147}$

c) $x^3\sqrt{x}$

$= \sqrt{(x^3)^2\, x}$

$= \sqrt{x^6 \cdot x}$

$= \sqrt{x^7}$

d) $-2x\sqrt{6x}$

$= -\sqrt{2^2 x^2\, 6x}$

$= -\sqrt{4 \cdot 6\, x^2 \cdot x}$

$= -\sqrt{24\, x^3}$

e) $2\sqrt[3]{7}$

$= \sqrt[3]{2^3 \times 7}$

$= \sqrt[3]{8 \times 7}$

$= \sqrt[3]{56}$

f) $3a^2b\,\sqrt[3]{b^2c}$

$= \sqrt[3]{3^3(a^2)^3\, b^3\, b^2 c}$

$= \sqrt[3]{27a^6\, b^5 c}$

g) $\frac{3x^2y}{5}\sqrt[3]{2xy^2}$

$= \sqrt[3]{\dfrac{3^3(x^2)^3y^3\, 2xy^2}{5^3}}$

$= \sqrt[3]{\dfrac{27 \cdot 2\, x^6 x\, y^3 y^2}{125}}$

$= \sqrt[3]{\dfrac{54\, x^7 y^5}{125}}$

1.5 – Adding and Subtracting Radical Expressions

Like Radicals

'Like Radicals' work very similarly to 'Like Terms'.

Simplify: $3x + 2x$ $= 5x$

Simplify: $3\sqrt{2} + 2\sqrt{2}$ $= 5\sqrt{2}$

Like radicals have the same index and same radicand

Steps for adding & subtracting like radicals:

① check if radicals are 'like'. If not, don't do any simplifying

② If yes, add or subtract coefficients, leave radicals the same

Example 1 – Simplify

a) $7\sqrt{3} - 2\sqrt{3}$

$5\sqrt{3}$

b) $-5\sqrt[3]{10} - 6\sqrt[3]{10}$

$-11\sqrt[3]{10}$

c) $4\sqrt[2]{2} - 5\sqrt[3]{2}$

index not the same, cannot simplify

d) $2\sqrt{75} + 3\sqrt{3}$

$2\sqrt{25 \cdot 3} + 3\sqrt{3}$

$2 \cdot 5\sqrt{3} + 3\sqrt{3}$

$10\sqrt{3} + 3\sqrt{3}$

$13\sqrt{3}$

e) $-\sqrt{27} + 3\sqrt{5} - \sqrt{80} - 2\sqrt{12}$

$= -\sqrt{9 \cdot 3} + 3\sqrt{5} - \sqrt{16 \cdot 5} - 2\sqrt{4 \cdot 3}$

$= -3\sqrt{3} + 3\sqrt{5} - 4\sqrt{5} - 4\sqrt{3}$

$= -7\sqrt{3} - \sqrt{5}$

a letter, could be any number

f) $\sqrt{9b} - 3\sqrt{16b}$, $b \geq 0$

$= 3\sqrt{b} - 3 \cdot 4\sqrt{b}$

$= 3\sqrt{b} - 12\sqrt{b}$

$= -9\sqrt{b}$, $b \geq 0$

In example f, why does b have to be greater than or equal to zero?

If it was negative, the radicands would become negative.

Example 2 – Simplify (assume variables are positive)

a) $\sqrt{27xy}+\sqrt{8xy}$

$= \sqrt{9\cdot 3xy} + \sqrt{4\cdot 2xy}$

$= 3\sqrt{3xy} + 2\sqrt{2xy}$

→ can't combine further since radican's are different

b) $4\sqrt[3]{16}+3\sqrt[3]{54}$

$= 4\sqrt[3]{8\cdot 2} + 3\sqrt[3]{27\cdot 2}$

$= 4\cdot 2\sqrt[3]{2} + 3\cdot 3\sqrt[3]{2}$

$= 8\sqrt[3]{2} + 9\sqrt[3]{2}$

$= 17\sqrt[3]{2}$

c) $3x\sqrt{63y}-5\sqrt{28x^2y}$

$= 3x\sqrt{9\cdot 7\cdot y} - 5\sqrt{4\cdot 7x^2y}$

$= 3\cdot 3x\sqrt{7y} - 5\cdot 2x\sqrt{7y}$

$= 9x\sqrt{7y} - 10x\sqrt{7y}$

$= -x\sqrt{7y}$

d) $\frac{5}{2}\sqrt[3]{16x^4y^5}-xy\sqrt[3]{54xy^2}$

$= \frac{5}{2}\sqrt[3]{8\cdot 2x^3xy^3y^2} - xy\sqrt[3]{27\cdot 2xy^2}$

$= \frac{5}{\cancel{2}_1}\cdot\frac{\cancel{2}^1}{1}xy\sqrt[3]{2xy^2} - 3xy\sqrt[3]{2xy^2}$

$= 5xy\sqrt[3]{2xy^2} - 3xy\sqrt[3]{2xy^2}$

$= 2xy\sqrt[3]{2xy^2}$

multiplying radicals

Example 1 – Multiply $2\sqrt{5}\,(3\sqrt{5})$ Verify your answer:

$= 2\cdot\sqrt{5}\cdot 3\cdot\sqrt{5}$
$= 2\cdot 3\cdot\sqrt{5}\cdot\sqrt{5}$
$= 6\sqrt{5\cdot 5} = 6\sqrt{25} = 6\cdot 5 = 30$

$(2\sqrt{5})(3\sqrt{5})$
$(4.472)(6.708)$
$= 30$ ✓

To multiply radicals:

① Multiply coefficients
② Multiply radicands (if index is the same)
③ Simplify

In general: $(x\sqrt[n]{a})(y\sqrt[n]{b})$

$= xy\sqrt[n]{ab}$

where n is a natural (counting #s) number and x, y, a, b are real (all #s) numbers. If n is even, $a \geq 0$ and $b \geq 0$ positive numbers

Example 2 – Simplify:

a) $5\sqrt{3}\,(\sqrt{6})$

$= 5\sqrt{18}$
$= 5\sqrt{9\cdot 2}$
$= 5\cdot 3\sqrt{2}$
$= 15\sqrt{2}$

b) $2\sqrt{6}\,(4\sqrt{8})$

$= 8\sqrt{48}$
$= 8\sqrt{16\cdot 3}$
$= 8\cdot 4\sqrt{3}$
$= 32\sqrt{3}$

c) $-3\sqrt{2x}\,(4\sqrt{3x})\quad x \geq 0$

$= -12\sqrt{6x^2}$

$= -12x\sqrt{6}$

d) $-2\sqrt[3]{11}(4\sqrt[3]{2} - 3\sqrt[3]{3})$

$-8\sqrt[3]{22} + 6\sqrt[3]{33}$

e) $(4\sqrt{2} + 3)(\sqrt{7} - 5\sqrt{14})$

$= 4\sqrt{14} - 20\sqrt{28} + 3\sqrt{7} - 15\sqrt{14}$
$= 4\sqrt{14} - 20\sqrt{4\cdot 7} + 3\sqrt{7} - 15\sqrt{14}$
$= 4\sqrt{14} - 40\sqrt{7} + 3\sqrt{7} - 15\sqrt{14}$

$= -11\sqrt{14} - 37\sqrt{7}$

f) $(\sqrt[3]{x} + \sqrt[3]{y})(\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2})$

$= \sqrt[3]{x^3} - \sqrt[3]{x^2y} + \sqrt[3]{xy^2} + \sqrt[3]{x^2y} - \sqrt[3]{xy^2} + \sqrt[3]{y^3}$

$= x + y$

dividing radicals

Example 3 – Divide $\frac{6\sqrt{12}}{3\sqrt{6}}$

$= 2\sqrt{2}$

$= 2.828$

Verify your answer: $\frac{6\sqrt{12}}{3\sqrt{6}} = \frac{20.785}{7.348}$

$= 2.828$

To divide radicals:
1. Divide coefficients
2. Divide radicands (if index the same)
3. Simplify if possible

In general: $\frac{x\sqrt[n]{a}}{y\sqrt[n]{b}} = \frac{x}{y}\sqrt[n]{\frac{a}{b}}$

with the same stipulations as multiplying radicals.

also $y \neq 0$, $b \neq 0$ (so $b > 0$)

Example 4 – Simplify: a) $\frac{-24\sqrt[3]{14}}{8\sqrt[3]{2}}$ b) $\frac{2\sqrt{51}}{\sqrt{3}}$ c) $\frac{\sqrt{18x^3}}{\sqrt{3x}}, x > 0$

a) $= -3\sqrt[3]{7}$

b) $= 2\sqrt{17}$

c) $= \sqrt{6x^2}$

$= x\sqrt{6}$

1.6B – Rationalizing the Denominator

rationalizing the denominator

A final answer cannot have a radical in the denominator. Therefore, you may have to '**rationalize the denominator**' – a process that will eliminate the radical from the denominator without changing the value of the expression.

If the denominator is a radical *monomial*, multiply the numerator and denominator by that radical.

$\sqrt{5}\cdot\sqrt{5} = \sqrt{5^2} = \sqrt{25} = 5$

Example 1 – Rationalize:

a) $\frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{3\sqrt{5}}{5}$

b) $\sqrt{\frac{2}{7}} = \frac{\sqrt{2}}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}$

$= \frac{\sqrt{14}}{7}$

c) $6\sqrt{\frac{3}{4x}}, x > 0$

$= \frac{6\sqrt{3}}{\sqrt{4x}}$

$= \frac{6\sqrt{3}}{2\sqrt{x}}$

$= \frac{3\sqrt{3}}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}$

$= \frac{3\sqrt{3x}}{x}$

d) $\frac{6}{7\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{6\sqrt{5}}{7\cdot 5}$

$= \frac{6\sqrt{5}}{35}$

to undo a cube root must cube it

$\left(\sqrt[3]{6}\right)^{3} = 6$

e) $\frac{2\sqrt{5}}{3\sqrt[3]{6}}\cdot\frac{\sqrt[3]{6}\cdot\sqrt[3]{6}}{\sqrt[3]{6}\cdot\sqrt[3]{6}}$

3 of them since cube root

$= \frac{\cancel{2}^{1}\sqrt{5}\,\sqrt[3]{36}}{3\cdot\cancel{6}\,3}$ can't combine since different index

$= \frac{\sqrt{5}\,\sqrt[3]{36}}{9}$

f) $\sqrt[3]{\frac{2}{y}} = \frac{\sqrt[3]{2}}{\sqrt[3]{y}}\cdot\frac{\sqrt[3]{y}\cdot\sqrt[3]{y}}{\sqrt[3]{y}\cdot\sqrt[3]{y}}$

$= \frac{\sqrt[3]{2y^2}}{y}$

g) $\frac{2}{\sqrt{x+1}}\cdot\frac{\sqrt{x+1}}{\sqrt{x+1}}$

$= \frac{2\sqrt{x+1}}{x+1}$

If the denominator is a radical *binomial*, multiply the numerator & denominator by its ***conjugate.*** → the same two terms but with the sign changed

ie: conjugate of $(\sqrt{x}-2)$ is $(\sqrt{x}+2)$

conjugates always expand to a difference of squares

$(a+b)(a-b)=a^2-b^2$

Example 2 – Rationalize:

a) $\frac{3}{(\sqrt{x}-2)} \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$

$$= \frac{3\sqrt{x}+6}{(\sqrt{x})^2-2^2} \quad \text{or} \quad = \frac{\sqrt{3x}+6}{x+2\sqrt{x}-2\sqrt{x}-4}$$

$$= \frac{3\sqrt{x}+6}{x-4} \qquad = \frac{\sqrt{3x}+6}{x-4}$$

b) $\frac{2+\sqrt{2}}{(3\sqrt{5}-4)} \frac{(3\sqrt{5}+4)}{(3\sqrt{5}+4)}$

$$= \frac{6\sqrt{5}+8+3\sqrt{10}+4\sqrt{2}}{(3\sqrt{5})^2-4^2}$$

$\to 3\sqrt{5}\cdot 3\sqrt{5}-4^2 \to 9\cdot 5-16 \to 45-16 \to 29$

$$= \frac{6\sqrt{5}+3\sqrt{10}+4\sqrt{2}+8}{29}$$

c) $\frac{\sqrt{a}+\sqrt{2b}}{\sqrt{a}-\sqrt{2b}}$, $a, b \geq 0$

$$\frac{\sqrt{a}+\sqrt{2b}}{\sqrt{a}-\sqrt{2b}} \cdot \frac{\sqrt{a}+\sqrt{2b}}{\sqrt{a}+\sqrt{2b}}$$

$$= \frac{a+\sqrt{2ab}+\sqrt{2ab}+2b}{(\sqrt{a})^2-(\sqrt{2b})^2}$$

$$= \frac{a+2\sqrt{2ab}+2b}{a-2b}$$

(note: $a \neq 2b$)

1.7 – Radical Equations

Radical Equations are mathematical equations that include a radical, such as

$$2\sqrt{6x} - 1 = 11$$

If the index of the radical is even, there are restrictions on the variable: since it is not possible to find the square root of a negative number, the radicand cannot be negative.

Example 1 – Find the restriction on the variable:

a) $2\sqrt{6x} - 1 = 11$

$\frac{6x}{6} \geq \frac{0}{6}$

$x \geq 0$

b) $\sqrt{x+2} = 49$

$x + 2 \geq 0$
$-2 \quad -2$

$x \geq -2$

c) $7\sqrt{-2x+3} = 35$

$-2x + 3 \geq 0$
$-3 \quad -3$

$\frac{-2x}{-2} \geq \frac{-3}{-2}$

flip $= x \leq \frac{3}{2}$

d) $\sqrt{3x+4} = \sqrt{2x-4}$

$3x + 4 \geq 0$
$-4 \quad -4$

$\frac{3x}{3} \geq \frac{-4}{3}$

$x \geq \frac{-4}{3}$

$2x - 4 \geq 0$
$+4 \quad +4$

$\frac{2x}{2} \geq \frac{4}{2}$

$x \geq 2$

so

$x \geq 2$

Steps to solving radical equations:

1. Find the restrictions on the variable in the radicand (if the index is even). Remember, the radicand must be set to ≥ 0 and then solved (if you multiply or divide by a negative number to both sides, FLIP the inequality).
2. Get the radical all by itself on one side of the equation.
3. If the index is 2, square both sides (if index is 3, cube both sides, etc.) and then solve for the variable.
4. See if the solution is affected by the restriction.
5. Check the answer using the original equation to see if solutions are valid or extraneous.

Example 2 – Solve: a) $2\sqrt{6x} - 1 = 11$

$2\sqrt{6x} - 1 = 11$
$+1 \quad +1$

$\frac{2\sqrt{6x}}{2} = \frac{12}{2}$

$(\sqrt{6x})^2 = (6)^2$

$\frac{6x}{6} = \frac{36}{6}$

$x = 6$

restriction

$\frac{6x}{6} \geq \frac{0}{6}$

$x \geq 0$ ✓

check

put # in place of x

LS	RS
$2\sqrt{6x} - 1$	11
$2\sqrt{6(6)} - 1$	
$2\sqrt{36} - 1$	
$2 \cdot 6 - 1$	
$12 - 1$	
11	

✓

b) $-8 + \sqrt{\frac{3y}{5}} = 2$

$-8 + \sqrt{\frac{3y}{5}} = 2$
$+8 \quad +8$

$\left(\sqrt{\frac{3y}{5}}\right)^2 = (10)^2$

$5 \times \frac{3y}{5} = 100 \times 5$

$\frac{3y}{3} = \frac{500}{3}$

$y = \frac{500}{3}$

restriction

$\frac{3y}{5} \geq 0$

$y \geq 0$ ✓

check

RS	LS
$-8 + \sqrt{\frac{\cancel{3}\left(\frac{500}{\cancel{3}}\right)}{5}}$	2
$-8 + \sqrt{\frac{500}{5}}$	
$-8 + \sqrt{100}$	
$-8 + 10$	
2	

✓

restriction

$2x-3 \geq 0$

$+3 \quad +3$

$\frac{2x}{2} \geq \frac{3}{2}$

$x \geq \frac{3}{2}$ ✓

Note: If there is just a radical on one side and just a negative constant on the other there are no solutions

If two radicals get one on each side then square

Example 3 – Solve: a) $4+\sqrt{2x-3}=1$

$4+\sqrt{2x-3} = 1$

$-4 \quad\quad -4$

$(\sqrt{2x-3})^2 = (-3)^2$

$2x-3 = 9$

$+3 \quad +3$

$\frac{2x}{2} = \frac{12}{2}$

$x = 6$

check

rs	LS
$4+\sqrt{2(6)-3}$	1
$4+\sqrt{12-3}$	
$4+\sqrt{9}$	
$4+3$	
7	✗

fail ∴ extraneous

No solution $\emptyset$

b) $-2\sqrt{x-5} = 16$

restriction

$x-5 \geq 0$

$x \geq 5$

$\frac{-2\sqrt{x-5}}{-2} = \frac{16}{-2}$

$\sqrt{x-5} = -8$

No solution $\emptyset$

Example 4 – Solve:

a) $\sqrt{10x-7} = 3\sqrt{x}$

restrictions

$10x-7 \geq 0$

$+7 \quad +7$

$\frac{10x}{10} \geq \frac{7}{10}$

$x \geq \frac{7}{10}$

and

$x \geq 0$

so

$x \geq \frac{7}{10}$

solve

$(\sqrt{10x-7})^2 = (3\sqrt{x})^2$

$10x-7 = 9x$

$+7 \quad +7$

$10x = 9x+7$

$-9x \quad -9x$

$x = 7$

Check

RS	LS
$\sqrt{10(7)-7}$	$3\sqrt{7}$
$\sqrt{63}$	
$\sqrt{9\cdot 7}$	
$3\sqrt{7}$	

✓

b) $2\sqrt{x} = \sqrt{7x+6}$

restrictions

$x \geq 0$

and

$7x+6 \geq 0$

$-6 \quad -6$

$\frac{7x}{7} \geq \frac{-6}{7}$

$x \geq \frac{-6}{7}$

so

$x \geq 0$

solve

$(2\sqrt{x})^2 = (\sqrt{7x+6})^2$

$4x = 7x+6$

$-7x \quad -7x$

$\frac{-3x}{-3} = \frac{6}{-3}$

$x = -2$

does not fulfill restrictions so no solutions!

no solution $\emptyset$

Example 5 – Solve:

$x \geq -1$

a) $\sqrt{x+1} = x-1$

$(\sqrt{x+1})^2 = (x-1)^2$

$x+1 = (x-1)(x-1)$

$x+1 = x^2-x-x+1$

$x+1 = x^2-2x+1$

$-1 \quad\quad -1$

$x = x^2-2x$

$-x \quad\quad -x$

$0 = x^2-3x$ } factor

$0 = x(x-3)$

split the two factors

$x=0, \quad x-3=0$

$+3 \quad +3$

$x=3$

$x=0, 3$

Check → need 2 checks, one for each solution

$x=0$

LS	RS
$\sqrt{x+1}$	$x-1$
$\sqrt{0+1}$	$0-1$
$\sqrt{1}$	-1
1	

✗

$x=3$

LS	RS
$\sqrt{x+1}$	$x-1$
$\sqrt{3+1}$	$3-1$
$\sqrt{4}$	2
2	

✓

Only valid solution is $x=3$

b) $m-\sqrt{2m+3} = 6$

restrictions

$2m+3 \geq 0$

$-3 \quad -3$

$\frac{2m}{2} \geq \frac{-3}{2}$

$m = -\frac{3}{2}$

solve

$m-\sqrt{2m+3} = 6$

$(m-6)^2 = (\sqrt{2m+3})^2$

$(m-6)(m-6) = 2m+3$

$m^2-12m+36 = 2m+3$

$-2m-3 \quad\quad -2m-3$

factor $m^2-14m+33 = 0$

$(m-11)(m-3) = 0$

$m=11, \quad m=3$

Check

$m=11$

LS	RS
$m-\sqrt{2m+3}$	6
$11-\sqrt{2(11)+3}$	
$11-\sqrt{25}$	
$11-5$	
6	

✓

$m=3$

LS	RS
$3-\sqrt{2(3)+3}$	6
$3-\sqrt{9}$	
$3-3$	
0	

✗

Solution is $m=11$