Chapter 1 - Real Numbers

1.1 - Number Systems

In math, it is important to be able to discuss different types of numbers. Some are positive (4), some negative (-37) , some can be written as fractions (1/2), and some can't be written as fractions (π). Also, what about numbers that are neither really positive nor negative, like the number 0? They all must be classified into Sets.

Number Sets:

First are the Natural Numbers (a.k.a the counting numbers).

They start at "1" and go up one-at-a-time. $\{1, 2, 3, 4 ...\}$

Natural Numbers Symbol: $\mathbb{N}_1$

Before we can discuss the negatives, we must include 0. This inclusion of 0 creates a whole new set. Known as the Whole Numbers.

$\{0, 1, 2, 3, 4 ...\}$

Whole Number Symbol: $\mathbb{N}_0$ or $\mathbb{W}$

Now we can include the negative numbers. This new set of numbers, which houses negatives, zero, and positives is called the set of Integers.

$\{... -3, -2, -1, 0, 1, 2, 3 ...\}$

Integers Symbol: $\mathbb{Z}$

Notice that until now, every number we have observed goes up or down one-at-a-time. But obviously that is not always the case. The Rational Numbers can now be introduced. These are numbers that are either terminating, or repeating; they are numbers that can be written as fractions.

Rational Numbers Symbol: $\mathbb{Q}$

Rational Number Examples: $\{\frac{1}{3}, 0, -4, \frac{-5}{3}, 2.6\overline{7}, 2.65\overline{7}, 2.56\}$

Note A: fractions <u>can not</u> have denominators equal to 0. Eg. 4/0 = undefined = ϕ

Note B: fractions with denominators equal to 1 are often re-written without a denominator at all. Eg. $\frac{3}{1} = 3, -\frac{.4}{1} = -.4$

Note C: Both the numerator and denominator in a fraction can be integers.

Note D: There are an infinite number of numbers, but there are more rational numbers than there are integers, and more integers than natural! Why is that?

Some numbers are unable to be expressed as fractions because they do not terminate, or repeat. They are known as Irrational Numbers

Irrational Number Symbol: $\overline{Q}$

Irrational Number Examples: $\{\sqrt{3}, \pi, e, 5.5454789\ldots\}$

All of the numbers listed above, make up the ultimate set, known as the set of Real Numbers.

Real Number Symbol: $\mathbb{R}$

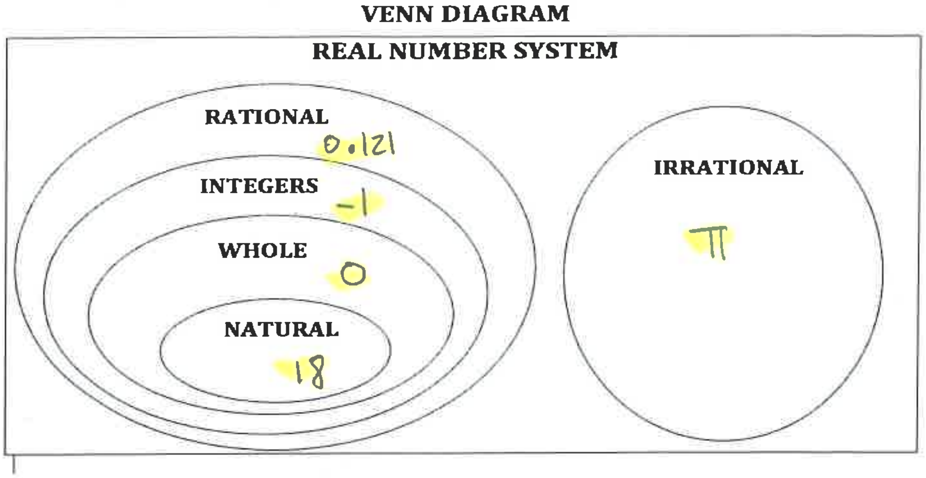
BONUS: All the numbers you can think of probably fall under the set of Real Numbers. If a number falls outside of the Real-Number Set, it is known as an Imaginary Number. Imaginary numbers, most importantly, allow us to solve negative roots! You won't be tested on this in grade 10.

Imaginary Number Symbol: $i = \sqrt{-1}$

Examples: $\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1} = 3 \times i = 3i$

$i^2 = (\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = -1$

Activity: Place the following numbers into the appropriate zones.
{0.121, 18, -1, 0, π}



1.2 - Greatest Common Factor (GCF) and Least Common Multiple (LCM)

The GCF and LCM are two important principles in mathematics. But to be able to understand and utilize them, we first must discuss Prime Numbers - the building blocks of our number system.

PRIME NUMBERS: A prime number is a whole number that has exactly two factors: 1 and itself. They have no *other* numbers that can divide it equally into wholes.

Most interestingly, every number is either itself a prime, or can reached by multiplying primes together.
Eg. **1** (exception), **2** (prime), 3 (prime), 4 (2x2), 5 (p), 6 (2x3), 7 (p), 8 (4x2 = 2x2x2), 9 (3x3), 10 (5x2) ...

Numbers that are made up by combining (multiplying) two primes together are called Composite Numbers. Composite numbers (4, 6, 8, 10, 25, 40, 1000) have factors other than just 1 or itself.

Note A: The whole numbers 0 and 1 are neither prime nor composite - they are exceptions to our definitions.

Note B: The whole number __2__ is the only even prime number.

Note C: There are an infinite number of primes! Some of the most interesting are 691, 78557, 1000000000000066600000000000001 (known as the devil's prime), and $2^{74207281}-1$ (which as of January 4, 2018 is the largest known prime; it has 23,249,425 digits!)

It is important to memorize some of the basic prime numbers, it will save you time in the future:

☆ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 ☆ ← memorize these

• 37, 41, 43, 47, 53, 59, 61, 67, 71, 79, 83, 89, 97 • ← just good to know

Divisibility Properties: Tricks that can help you factor composite numbers quickly.

Divisibility by 2: if the last digit is even (0, 2, 4, 6, 8).

$548 \div 2 = \boxed{274}$

Divisibility by 3: if the sum of its digits are divisible by 3.

$1147302 \div 3 = \boxed{382434}$

$\to 1+1+4+7+3+0+2 = 18 \div 3 = 6$ ✓

Divisibility by 4: if the last two digits are divisible by 4.

$548 \div 4 = \boxed{137}$

Divisibility by 5: if the last digit is a 0 or 5.

$365 \div 5 = \boxed{73}$

Divisibility by 6: if it is an even number that is divisible by 3.

$1147302 \div 6 = \boxed{191217}$

Divisibility by 9: if the sum of its digits are divisible by 9.

$963 \div 9 = \boxed{107}$

$\to 9+6+3 = 18 \div 9 = 2$ ✓

Divisibility by 10: if it ends in a 0.

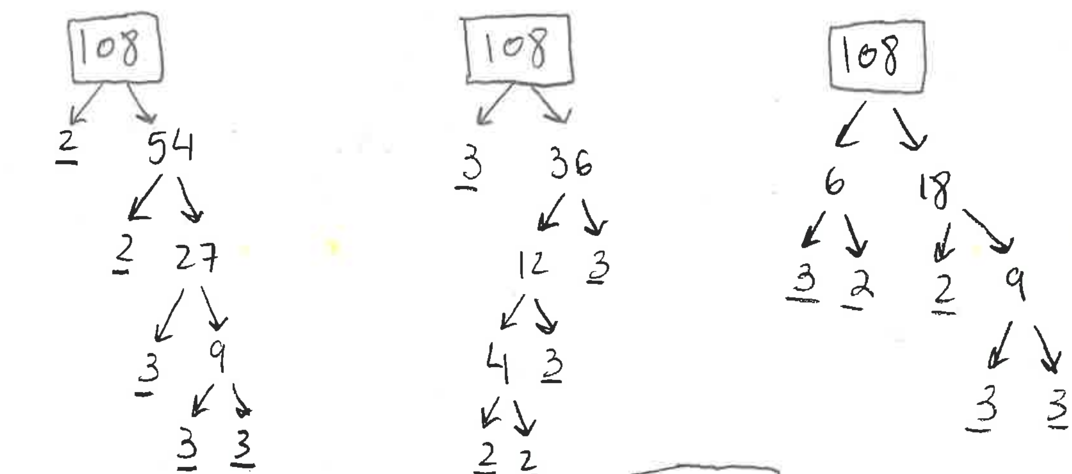
$12380 \div 10 = \boxed{1238}$

Finding Prime Factors of Composite Numbers (Prime Factor Decomposition)

Method 1: Factor Tree

1) Start by writing the composite number you want to factor at the top line.
2) Split the composite into two known factors, and write them on the line below (if you don't know big ones, start with the smallest ones)
3) If the subsequent factor is composite, do step 2 again. If it is prime, stop, as it can't be factored further.
4) Once all factors have reached the prime stage, collect them, and rewrite.

Example: Find the prime factors for 108 using the factor tree method.



Prime factors: $2 \times 2 \times 3 \times 3 \times 3 = \boxed{2^2 \times 3^3}$

Notice that no matter the method, the prime factors end up the same.

Method 2: Continuous Division

1) Write the original composite number within the division-L. To the left, write a known prime factor.
2) Below, write the dividend (what you get after dividing out your prime), in a new division-L. To the left, write another prime factor that can divide this new number.
3) Once the dividend is a prime, you have finished factoring. Collect all primes (including the last one), and then you're done.

Eg. Find all prime factors of 960.

3 | 960
2 | 320
2 | 160
2 | 80
2 | 40
2 | 20
2 | 10
5 | 5
1

Prime factors: $\boxed{3^1 \times 2^6 \times 5^1}$

When comparing two or more numbers, we can find their GREATEST COMMON FACTOR (GCF) or the LOWEST COMMON MULTIPLE (LCM).

Greatest Common Factor

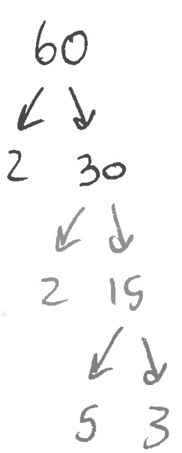
Note: The word "factor" implies the answer will be smaller than the original composites.

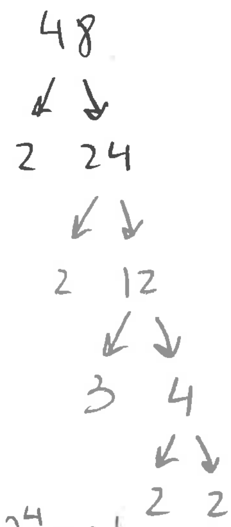
The GCF is the largest number that divides each of the given numbers exactly.

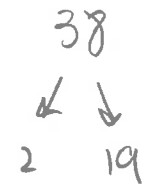
Steps to finding the GCF.

1. Find the prime factors of each number (using either method shown above)
2. List each common factor the least number of times it appears in any one number. (In other words: just collect the overlapping primes)
3. Multiply the collected factors to find your GCF.

Example 1) Find the GCF of 60, 48, 38







$60: 2^2 \times 5^1 \times 3^1$

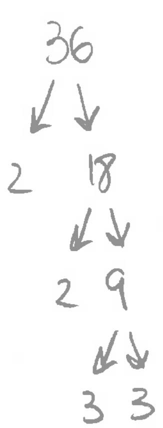
$48: 2^4 \times 3^1$

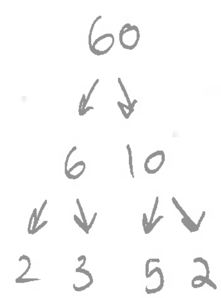
smallest

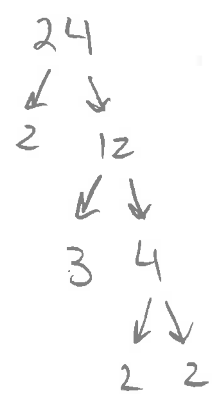
$38: 2^1 \times 19^1$

$\boxed{GCF = 2^1}$ as it is the only complete overlap!

Example 2) Find the GCF of 36, 60, 24







$36: 2^2 \times 3^2$

$60: 2^2 \times 3^1 \times 5^1$

$24: 2^3 \times 3^1$

The overlapping primes are 2^2 and 3^1

$GCF = 2^2 \times 3^1 = 4 \times 3 = \boxed{12}$

Lowest Common Multiple

Note: The word "multiple" implies the answer will be larger than the original composites.

The LCM is the smallest common non-zero multiple of two or more whole numbers, or the smallest number that is divisible by all the numbers. (In other words: it is the lowest combined value, that has the original numbers as its factors)

Finding the LCM:

1) Use a factor tree to find the prime factors of your given numbers.
2) Select the primes that occur the greatest number of times in any one factor. (**a.k.a.** Choose the most powerful of each unique prime **a.k.a.** "*all the best*")
3) The LCM is the product of the selected primes.

Example 3) Find the LCM of 60, 48, 38

From example 1, we know that the prime factors of our numbers are:

60: $2^2 \times \underline{3^1} \times \underline{5^1}$

48: $\underline{2^4} \times 3^1$

38: $2^1 \times \underline{19^1}$

- To find the LCM, collect every unique prime, selecting the greatest power whenever there is an overlap.
- Our unique prime factors are: 2, 3, 5, 19. The highest power of each is: $2^4 \times 3^1 \times 5^1 \times 19^1 = \boxed{4560}$

Example 4) Find the LCM of 36, 60, 24

From example 2, we know that the prime factors are:

$36: (2^2)(3^2)$
$60: (2^2)(3^1)(5^1)$
$24: (2^3)(3^1)$

- Unique factors are 2, 3, 5
- Highest power of each: $(2^3)(3^2)(5^1) = \boxed{360}$

1.3 - Squares and Square Roots

In other words

To "square" a number means to raise the number to the second power (IOW: multiplying a number by itself).

Eg) $3 \times 3 = 3^2 = 9$, $4^2 = 4 \times 4 = 16$, $25 = 5 \times 5 = 5^2$

In math, for every action there is an equal and opposite action. So if we can easily "square" two numbers to find its product, we can also SQUARE ROOT a number to find its identical factors.

The symbol $\sqrt[2]{}$ (called a radical sign, with index 2) is used to indicate square roots, but we can use the symbol $\sqrt{}$ as the common shorthand. (Notice, that the index "2" is erased)

Note: You will also use these symbols in the future to discuss higher powered roots.

$\sqrt[3]{}$ = cube root

$\sqrt[5]{}$ = fifth root

$\sqrt[4]{}$ = forth root

PERFECT SQUARES: Numbers with rational square roots (no crazy decimals)

It is important to have a base-memorization-level of the most common whole number perfect squares:

Perfect Square	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
Square root	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

If perfect whole numbers can easily be rooted, so can the subsequent perfect rational numbers.

If $\sqrt{9} = 3$ and $\sqrt{16} = 4$, then $\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$

Eg. Simplify $\sqrt{\frac{25}{30}}$

$\sqrt{\frac{25}{30}} = \frac{\sqrt{25}}{\sqrt{30}}$ ← perfect Square (solve) / ← imperfect (leave)

$= \frac{5}{\sqrt{30}}$

Solving decimal roots: When solving roots of decimals, it is best to convert the decimal to a fraction, and proceeding just like the above examples.

Eg. $\sqrt{.04} = \sqrt{\frac{4}{100}} = \frac{\sqrt{4}}{\sqrt{100}} = \boxed{\frac{2}{10}}$

$\sqrt{.4} = \sqrt{\frac{4}{10}} = \frac{\sqrt{4}}{\sqrt{10}} = \boxed{\frac{2}{\sqrt{10}}}$

Finding Square Roots Without a Calculator

We can use the factor tree to determine the square roots of perfect whole numbers.

Eg 2) Determine the square root of 225

225

25 9

5 5 3 3

$$\therefore \sqrt{225} = \sqrt{5\times5\times3\times3}$$
$$= \sqrt{5\times5} \times \sqrt{3\times3}$$
$$= \sqrt{5^2} \times \sqrt{3^2}$$
$$= 5 \times 3$$
$$= \boxed{15}$$

Eg 3) Determine the square root of 400

400

4 100

2 2 10 10

5 2 5 2

$$\therefore \sqrt{400} = \sqrt{2\times2\times2\times2\times5\times5}$$
$$= \sqrt{2^2\times2^2\times5^2}$$
$$= \sqrt{2^2}\times\sqrt{2^2}\times\sqrt{5^2}$$
$$= 2\times2\times5$$
$$= \boxed{20}$$

ALSO

$$\sqrt{400} = \sqrt{4}\times\sqrt{100}$$
$$= 2\times10$$
$$= \boxed{20}$$

Note: For whole numbers, $\sqrt[2]{a^2} = \sqrt{a\times a} = a$

Cubes and Cube Roots

To "cube" a number means to raise the number to the third power, or to multiply the number by itself three times.

For example: $3^3 = 3 \times 3 \times 3 = 27$ $\quad 4^3 = 4 \times 4 \times 4 = 64$ $\quad 125 = 5 \times 5 \times 5 = 5^3$

These identical factors is called the Cube root of a number. As mentioned earlier, it has the symbol: $\sqrt[3]{}$ (the little 3 is known as the index).

Here is a list of a few cube roots that should be memorized.

Perfect Cubes	0	1	8	27	64	125	216	343	1000
Cube Roots	0	1	2	3	4	5	6	7	10

Eg 4) Determine the cube root of $\frac{64}{216}$

$$\sqrt[3]{\frac{64}{216}} = \frac{\sqrt[3]{64}}{\sqrt[3]{216}} = \frac{\sqrt[3]{4^3}}{\sqrt[3]{6^3}} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

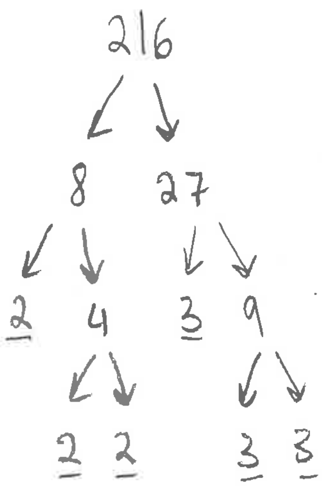
Note: *In the expression* $\sqrt[k]{a}$, *we call 'k' the index and 'a' the radicand.*
We assume $k \geq 2$. *If the index is not written, the expression is assumed to be a square root, with* $k = 2$.

Eg. $\sqrt[6]{64} = 2$ *because* $2 \times 2 \times 2 \times 2 \times 2 \times 2$ *or* $2^6 = 64$

Index $= 6$

Radicand $= 64$

Eg Find the cube root 216



$$\therefore \sqrt[3]{216} = \sqrt[3]{2\times2\times2\times3\times3\times3}$$
$$= \sqrt[3]{2\times2\times2} \times \sqrt[3]{3\times3\times3}$$
$$= \sqrt[3]{2^3} \times \sqrt[3]{3^3}$$
$$= 2 \times 3$$
$$= \boxed{6}$$

Note: *For whole numbers*, $\sqrt[3]{a^3} = \sqrt[3]{a \times a \times a} = a$

On that note, $\sqrt[k]{a^k} = a$ *(as the power and root [both k] cancel eachother out)*

1.4 - Rational and Irrational Numbers

Perfect-powers are numbers that have rational roots.

Eg.

$$\sqrt{49} = 7 \text{ or } \sqrt[3]{125} = 5 \text{ or } \sqrt{\frac{25}{81}} = \frac{\sqrt{25}}{\sqrt{81}} = \frac{5}{9} \text{ or } \sqrt[3]{\frac{8}{343}} = \frac{\sqrt[3]{8}}{\sqrt[3]{343}} = \frac{2}{7}$$

But this is actually very rare, as most numbers, when rooted, form irrational numbers (numbers that go on forever, and do not repeat):

$\pi = 3.14159265\ldots$

$e = 2.71828182\ldots$ $\qquad \sqrt{\text{any non-perfect number}}$

$\sqrt{10} = 3.16227\ldots$

$\sqrt{2} = 1.414213\ldots$

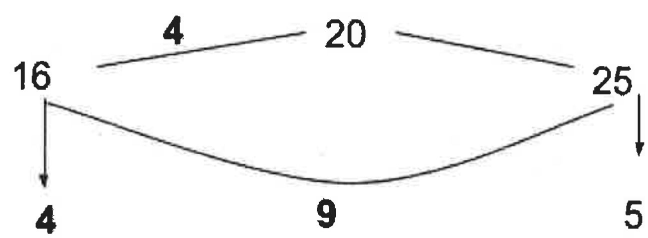
As you move up in math, it becomes simpler to leave rooted numbers in radical form. To gain an increased number sense, we must be able to approximate irrational roots without the use of a calculator.

Eg 1) Approximate $\sqrt{20}$ to one decimal place

Steps: 1) Ask yourself what perfect squares are just less and more than 20: 16 & 25

2) As the $\sqrt{16} = 4$ and the $\sqrt{25} = 5$ we know that $\sqrt{20}$ sits somewhere between 4 and 5.

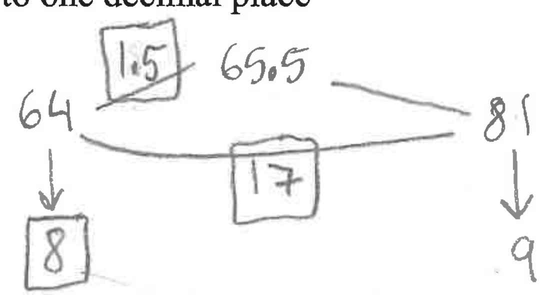
3) Next we must approximate the decimal place: To do this, we first see how far away our given root is from its lesser perfect square (this will be our numerator), and how far away the two perfect squares are from each other (this will our denominator).



Therefore the fraction is $\frac{4}{9} \cong .45$

4) Combine the whole number from 2), with the decimal from 3) to form your answer: **4.45** $\sqrt{20} = 4.4721...$

Eg 2) Approximate $\sqrt{65.5}$ to one decimal place



$\therefore \sqrt{65.5} \approx 8\frac{1.5}{17}$

approx.

$\frac{1.5}{17} \approx \frac{2}{18} = \frac{1}{9} = .\overline{111} \Rightarrow \boxed{8.1}$

actual: $\boxed{8.093...}$

This process for approximating roots is quick but imperfect. On a quiz/test I will be looking for your number-sense. Any answers that are within a ballpark of appropriateness will be accepted.

1.5 - Exponential Notation

An exponent tells how many times the bases is used as a factor. In the statement $b \times b \times b \times b \times b = b^5$, the exponent is 5 and the base is b. The statement is read "b to the fifth power" or "b to the power of 5."

Exponents are one of the fundamental operators in mathematics. And make up the E in GEMA (or BEDMAS). As noted in the order-of-operations acronym, it is a very "powerful" operator and is tightly connected to its base (more so than multiplication).

You will see exponents used in many ways, here are some of the few:

- $4^3 = 4 \times 4 \times 4 = \boxed{64}$
- $(2y)^4 = (2y)(2y)(2y)(2y) = 2 \times 2 \times 2 \times 2 \times y \times y \times y \times y = (16)(y^4) = \boxed{16y^4}$
- $-5x^3 = -5 \times x \times x \times x$

(notice the exponent acts only on x when there are no brackets)

One and Zero as Exponents

Rule: Anything to the power of 1, is itself.

Eg.

$4^1 = 4$, $b = b^1$, $\left(\frac{1}{4}\right)^1 = \frac{1}{4}$

Rule: Anything to the power of 0 is 1. We will discuss why later in this section.

Eg.

$4^0 = 1$ $\quad b^0 = 1$ $\quad \left(\frac{1}{4}\right)^0 = 1$

Note: As you continue in math, you will observe many contradictions. One of them is 0^0. Is it 0 because anything times 0 is 0? Or is it 1 because anything to the power of 0 is 1? In fact, it's neither, and the correct answer is that 0^0 = undefined.

Exponent Rules

The Product Rule

Consider:

$$2^4 \times 2^2 = (2\times 2\times 2\times 2)\,(2\times 2) = (2\times 2\times 2\times 2\times 2\times 2) = 2^6 = 64$$

$$y^3 \times y^3 = (y\times y\times y)\times(y\times y\times y) = (y\times y\times y\times y\times y\times y) = y^6$$

$$Therefore\; y^3 \times y^3 = y^{3+3} = y^6$$

To find the product of two (or more) numbers with the Same base,
just add the exponents together. Leave the base the same!

General Equation ($b \neq 0$) :

$$\boxed{b^m \times b^n = b^{(m+n)}}$$

Examples:

E $\quad (-1)^6 \times (-1)^4 = (-1)^{10} = \boxed{1}$

M $\quad \left(-\frac{3}{4}\right) \times \left(\frac{-3}{4}\right)^2 = \left(\frac{-3}{4}\right)^1 \times \left(\frac{-3}{4}\right)^2 = \left(\frac{-3}{4}\right)^3 = \boxed{\frac{-27}{64}}$

H $\quad 3^{2x} \times 9^{6x} = 3^{2x} \times (3^2)^{6x} = 3^{2x} \times 3^{12x} = \boxed{3^{14x}}$

E $\quad 4^2 \times -4^3 = 4^2 \times 4^3 \times -1 = 4^5 \times -1 = \boxed{-4^5}$

The Quotient Rule

Consider:

$$\frac{27}{9}=\frac{3^3}{3^2}=\left(\frac{3\times 3\times 3}{3\times 3}\right)=\left(\frac{\cancel{3}\times\cancel{3}\times 3}{\cancel{3}\times\cancel{3}}\right)=3^1=3$$

$$\frac{w^5}{w^3}=\frac{w\times w\times w\times w\times w}{w\times w\times w}=\frac{\cancel{w}\times\cancel{w}\times\cancel{w}\times w\times w}{\cancel{w}\times\cancel{w}\times\cancel{w}}=w^2$$

$$Therefore\ \frac{w^5}{w^3}=w^{5-3}=w^2$$

To find the quotient of two (or more) numbers with the Same Base, just Subtract the exponents. The base does not change!

Remember:

× of bases → + exponents

÷ of bases → − exponents

General Equation ($b \neq 0$) : $\frac{b^m}{b^n} = \boxed{b^{m-n}}$

Examples:

- $\frac{(-2)^{11}}{(-2)^8} = (-2)^3 = -2 \times -2 \times -2 = \boxed{-8}$
- $\frac{\left(\frac{2}{4}\right)^5}{\left(\frac{1}{2}\right)^3} = \frac{\left(\frac{1}{2}\right)^5}{\left(\frac{1}{2}\right)^3} = \left(\frac{1}{2}\right)^{5-3} = \left(\frac{1}{2}\right)^2 = \frac{1^2}{2^2} = \boxed{\frac{1}{4}}$
- $\frac{5^3}{5^5} = 5^{3-5} = 5^{-2} = \frac{1}{5^2} = \boxed{\frac{1}{25}}$

We are now able to explore why $x^0=1$.
Consider:

$\frac{5}{5}=1$ and $\frac{33}{33}=1$ So $\frac{y}{y}=1$

$\therefore \frac{8}{8}=\frac{2^3}{2^3}=2^{3-3}=2^0=1$

General: $\frac{x^a}{x^a}=x^{a-a}=x^0=1$

The Power Rule

Consider the expression $(6^4)^3=6^4\times 6^4\times 6^4$. By the product rule

$$6^4\times 6^4\times 6^4=6^{4+4+4}=6^{12}$$

Whenever you have repeated addition of the same number, you can use multiplication instead. So, we can simplify this problem by multiplying the exponents together: $(6^4)^3=6^{4\times 3}=6^{12}$.

When finding the final exponent of a number to a power, to a power: simply multiply the exponents together.

General Equation: $(b^m)^n=b^{m\times n}=\boxed{b^{mn}}$

Examples:

- $(3^4)^4=3^{4\times 4}=3^{16}=43046721$ NOT $(3^4)^4\neq 3^8$
- $(7^6)^0=7^{6\times 0}=7^0=1$
- $\left(\left(\frac{3}{8}\right)^5\right)^2=\left(\frac{3}{8}\right)^{5\times 2}=\left(\frac{3}{8}\right)^{10}$

Raising a Product to a Power

An expression that itself contains a product, such as $(2y)^4 = (2y)(2y)(2y)(2y)$, can be re-written as $(2 \times 2 \times 2 \times 2)(y \times y \times y \times y) = 2^4 \times y^4 = 16y^4$. Notice: $(2y)^4 = 2^4 y^4 = 16y^4$

General Equation: $(ab)^n = (a \times b)^n = a^n \times b^n$

Examples:

$(4x)^2 = (4x)(4x) = 16x^2$

$(-5k^3h^7)^3 = (-5^3)(k^{3\times3})(h^{7\times3}) = \boxed{-125k^9h^{21}}$

Raising a Fraction to a Power

An expression such as $(\frac{3}{4})^3$ means $\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}$. Multiplying the fractions together gives $\frac{3\times3\times3}{4\times4\times4} = \frac{3^3}{4^3} = \frac{27}{64}$. Notice: $\left(\frac{3}{4}\right)^3 = 3^3/4^3 = 27/64$

General Equation ($y \neq 0$) : $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

Examples:

- $\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2} = \frac{x^2}{9}$

- $\left(\frac{b^3}{5a}\right)^6 = \frac{b^{3\times6}}{5^{a\times6}} = \frac{b^{18}}{5^{6a}}$

Negative Exponents

Negative exponents can be a bit confusing because until now we have only ever seen negatives used with coefficients. When negatives are used with exponents, however, they turn your base into a fraction -- the numerator becomes the denominator, and the denominator becomes the numerator!

Consider:

$$\frac{2^4}{2^7}=\frac{2\times2\times2\times2}{2\times2\times2\times2\times2\times2\times2}=\frac{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}}{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}\times2\times2\times2}=\frac{1}{2\times2\times2}=\frac{1}{2^3}=\frac{1}{8}$$

We are able to solve this problem quickly now that we know the quotient rule...

$$\frac{2^4}{2^7}=2^{4-7}=2^{-3}=\frac{1}{8}$$

Therefore $\frac{1}{2^3}=2^{-3}$. *Notice that* $2^3=8$ *and that* $2^{-3}=\frac{1}{8}=\frac{1}{2^3}$

General Equation ($b\neq 0$)

$$\boxed{b^{-n}=\frac{1}{b^n}}$$

Examples:

- $\frac{1}{25}=\frac{1}{5^3}=5^{-3}$

- $(3x)^{-3}=\frac{1}{(3x)^3}=\frac{1}{3^3\times x^3}=\frac{1}{27x^3}$

Changing from Negative to Positive Exponents

Numbers with negative exponents in the numerator, can be rewritten such that they become positive but appear in the denominator. This is also true in reverse.

Consider:

$$4^{-3} = \frac{1}{4^3} = \frac{1}{64} \qquad \textit{ALSO} \qquad \frac{1}{5^{-3}} = 5^{-(-3)} = 5^3 = 125$$

General Equation (doesn't hold for numbers=0):

$$\frac{a^c}{b^d} = \frac{b^{-d}}{a^{-c}} \qquad \text{AND} \qquad \left(\frac{a}{b}\right)^c = \left(\frac{b}{a}\right)^{-c}$$

Examples

- $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \boxed{\frac{16}{9}}$

- $\frac{(5^{-2})}{(7^{-3})(2^2)} = \frac{(7^3)}{(5^2)(2^2)} = \frac{343}{25 \times 4} = \boxed{\frac{343}{100}}$

Rational Exponents: $a^{1/n}$ and $a^{m/n}$

Rational exponents mean something different than the rational coefficients that we are used to seeing. If __4x__ means multiplying something by __4__, __$\frac{1}{4}x$__ is the opposite, and means dividing something by __4__. Rational exponents do the same thing: If x^2 means "x-squared" then $x^{1/2}$ mean the "__Square root of x__"

Lets see how this works:

$$\sqrt{3}\times\sqrt{3}=\left(\sqrt{3}\right)^2=3$$

And since

$\sqrt{3}=3^{\frac{1}{2}}$ *we can rewrite the above example such that*:

$$3^{\frac{1}{2}}\times 3^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{2}}=3^1=3$$

General Equation (both *b* and *n* are positive):

$$\sqrt[n]{b} = \sqrt[n]{b^1} = b^{1/n}$$

Example:

makes things smaller → $\sqrt[5]{3} = 3^{1/5}$ ← makes things smaller

We can generalize this formula to deal with more complicated fractions (not just that have a 1 for the numerator)

General Equation:

$$b^{m/n} = \sqrt[n]{b^m}$$

Consider the following:

$$10^{5/3} = \sqrt[3]{10^5} = 46.415\ldots$$

Examples:

- $\left(\sqrt[4]{16}\right)^3 = 16^{3/4} = \left(2^4\right)^{3/4} = 2^{4 \times 3/4} = 2^3 = \boxed{8}$

- $\dfrac{\sqrt[4]{81}}{\left(\sqrt[4]{2}\right)^{-8}} = \left(\sqrt[4]{81}\right)\left(\sqrt[4]{2}\right)^8 = \left(\sqrt[4]{3^4}\right)\left(\sqrt[4]{2^8}\right)$

$= \left(3^{4/4}\right)\left(2^{8/4}\right) = \left(3^1\right)\left(2^2\right) = 3 \times 4$

$= \boxed{12}$

- $\sqrt[2]{4^3} = 4^{3/2}$ $\xrightarrow{\text{cube first}}$ $64^{1/2} = \boxed{8}$

  $4^{3/2}$ $\xrightarrow{\text{square root first}}$ $2^3 = \boxed{8}$

Summary of Exponent Rules

For any integers m and n:		
Exponent of 1	$a^1 = a$	$3^1 = 3$
Exponent of 0	$a^0 = 1, \quad a \neq 0$	$(-5)^0 = 1$
Product Rule	$a^m \times a^n = a^{m+n}, \quad a \neq 0$	$2^3 \times 2^4 = 2^{3+4} = 2^7$
Quotient Rule	$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	$\frac{3^5}{3^3} = 3^{5-3} = 3^2$
Power Rules	$(a^m)^n = a^{m \times n}$ $(ab)^n = a^n \times b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$(2^3)^4 = 2^{3 \times 4} = 2^{12}$ $(2x)^3 = 2^3 \times x^3$ $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$
Negative Exponents	$a^{-n} = \frac{1}{a^n}$ $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$	$2^{-3} = \frac{1}{2^3}$ $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$ $\frac{2^{-3}}{3^{-4}} = \frac{3^4}{2^3}$
Rational Exponents	$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\sqrt[n]{a^m} = a^{\frac{m}{n}}$	$\sqrt[3]{5} = 5^{\frac{1}{3}}$ $\sqrt[4]{5^3} = 5^{\frac{3}{4}}$

1.6 - Irrational Numbers

Until now, many of the problems we have done have involved perfect powers (numbers that have equal roots like 4, 16, 27, 81, etc...). As mentioned, most numbers are not like this, but that does not mean they can't be simplified. We will use the product rule to help us do this.

General Equation:

$$\sqrt[n]{AB} = \sqrt[n]{A \times B} = \sqrt[n]{A} \times \sqrt[n]{B}$$

To maximize the effectiveness of this rule, you must be able to recognize hidden perfect roots.

Consider:

$$\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5 \times \sqrt{3} = 5\sqrt{3}$$

The objective here is to:

1) Recognize any perfect numbers that are factors of the original value (there may be more than one)
2) Split up the radical product so at least one of the factors is a perfect number.
3) Use the product rule to separate the radical
4) Simplify the perfect radical, leaving the other radical as is
5) Combine your terms (the simplified term becomes the coefficient for the remaining radical)

Examples

- $\sqrt{800} = \sqrt{100 \times 4 \times 2} = \sqrt{100} \times \sqrt{4} \times \sqrt{2}$
  $= 10 \times 2 \times \sqrt{2} = \boxed{20\sqrt{2}}$
- $\sqrt{3} \times \sqrt{20} = \sqrt{60} = \sqrt{4 \times 15} = \sqrt{4} \times \sqrt{15} = \boxed{2\sqrt{15}}$
  $\sqrt{3} \times \sqrt{5} \times \sqrt{4} = \sqrt{15} \times 2 = \boxed{2\sqrt{15}}$
- $\sqrt[3]{108} = \sqrt[3]{27 \times 4} = \sqrt[3]{27} \times \sqrt[3]{4} = \boxed{3\sqrt[3]{4}}$

Entire Root vs. Mixed Root

Just like how there are improper fractions and mixed fractions that denote two different ways of writing the same term; there are also entire roots and mixed roots, two different ways of writing roots.

Mixed Root		Entire Root
$5\sqrt[3]{4}$	$=$	$\sqrt[3]{500}$

Note: Any mixed root can be written as an entire root. But not necessarily the converse is true.

We have already learned how to rewrite entire roots as mixed roots, here are the steps to convert mixed roots to entire roots.

1) Observe the index of the radical, then observe the coefficient
2) Draw one long radical symbol, under-which you will write the coefficient, multiplied by itself, the index-number of times, as well as original radicand.
3) Rewrite the radical with the original index and the new product.

Examples:

- $5\sqrt[3]{4} = \sqrt[3]{5 \times 5 \times 5 \times 4} = \sqrt[3]{125 \times 4} = \boxed{\sqrt[3]{500}}$
- $2\sqrt[4]{3} = \sqrt[4]{2^4 \times 3} = \sqrt[4]{2 \times 2 \times 2 \times 2 \times 3}$
  $= \sqrt[4]{16 \times 3} = \boxed{\sqrt[4]{48}}$
- $10\sqrt{7} = 10\sqrt[2]{7} = \sqrt[2]{10^2 \times 7} = \sqrt{10 \times 10 \times 7}$
  $= \boxed{\sqrt{700}}$

Notes: