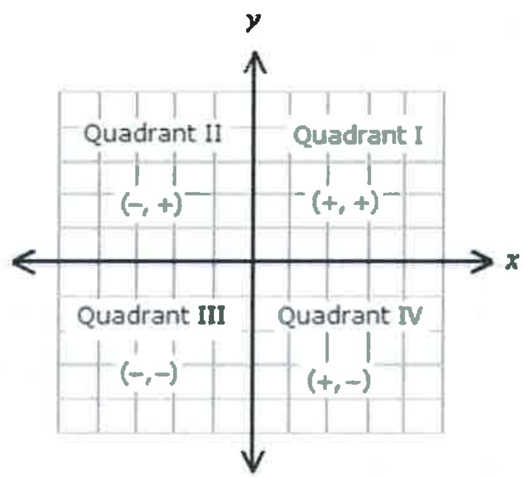
Chapter 3 - Relations and Functions

3.1 - Relations

Graphs are a fundamental aspect of mathematics and play an integral part of the secondary school curriculum. They may feel overwhelming at first, but the longer you spend working with them the easier they will become.

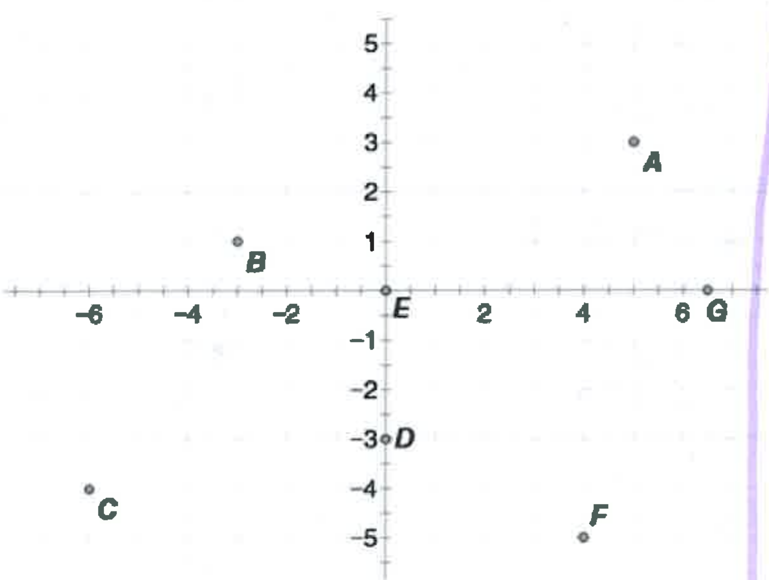
Coordinate System

The coordinate system uses a pair of numbers (coordinates) to uniquely determine the position of geometric elements. "Ordered pairs" are made up of one x and one y value, they are written as (x,y). A unique ordered pair will correspond to a unique point on a graph.



Note that the x-coordinate is always written first!

State the ordered pair for the corresponding letters.



A: (5,3) = 5 Right, 3 Up

B: (-3,1) = 3 Left, 1 Up

C: (-6,-4) = 6 Left, 4 Down

D: (0,-3) = 0 R/L, 3 Down

E: (0,0) = located at Origin

F: (4, -5) = 4 Right, 5 Down

G: (6.5,0) = 6.5 Right, 0 U/D

Relations

Relations are a set (or collection) of ordered pairs (x,y).

The set of all the x-values (the spread if you will), is called the **Domain**

The set of all the y-values (again, the full spread of them), is called the **Range**

Relations are often in a form where we "solve for y" like in the formulas:

$$y = mx + b \quad OR \quad y = ax^2 + bx + c \quad OR \quad y = a(x-h)^2 + k$$

Because of this, we often get to input an x-value of our choice, in an attempt to find out what y is. The x-value is thus known as the **independent variable**, and the y-value is known as the **dependent variable**. This is because the y-value is "dependent" on our particular equation, but x-value, which we get to choose, does not change based on our equation; it is "independent" from in.

For example:

Input (independent) x	Relation (formula) $y = 3x - 1$	Output (dependent) y
0	$y = 3(0) - 1$ $y = -1$	***-1***
1	$y = 3(1) - 1$ $y = 2$	***2***
-2	$y = 3(-2) - 1$ $y = -7$	***-7***
5	$y = 3(5) - 1$ $y = 14$	***14***

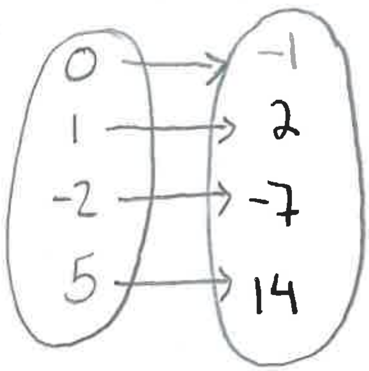
The x and y values represent our four chosen solutions to our relation $y = 3x - 1$.

The solutions can be presented in several ways:
As ordered pairs: (0,-1), (1,2), (-2, -7), (5,14)

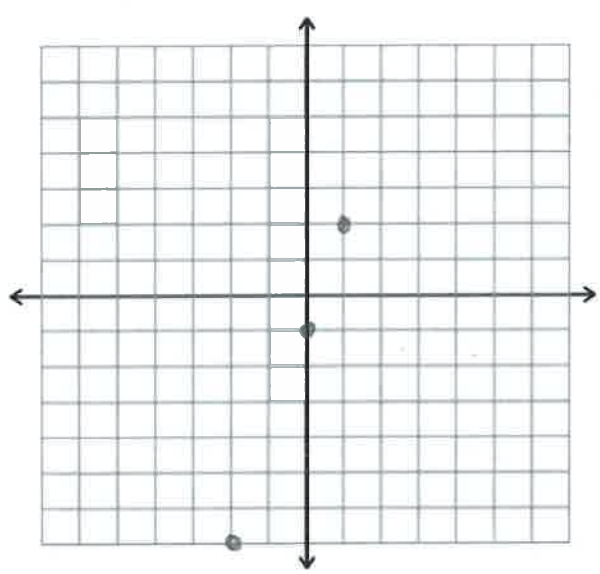
As a table of values:

x	y
0	-1
1	2
-2	-7
5	14

Using mapping notation:



As a graph



Determine the domain and range of the ordered pairs.

A={ (-1, -2), (3, -5), (-4, 2) }
B={ (3, -4), (-1, 0), (0, -2), (-3, -2) }
C={ (2, -1), (-1, 0), (-3, -3), (-1, -4) }

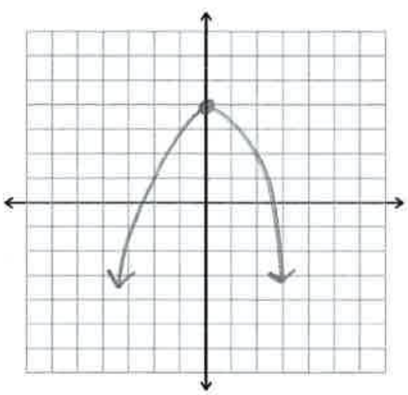
Domain of A = {-4, -1, 3}
Domain of B = {-3, -1, 0, 3}
Domain of C = {-3,-1,2}

Range of A = {-5, -2, 2}
Range of B = {-4, -2, 0} Notice we omit duplicates
Range of C = {-4, -3, -1, 0}

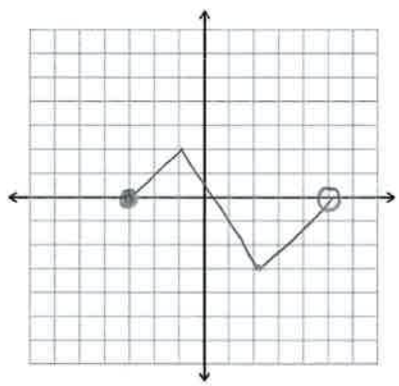
Domain of ABC = {-4, -3, -1, 0, 2, 3} Range of ABC = {-5, -4 , -3, -2, -1, 0, 2}

Determine the domain and range:

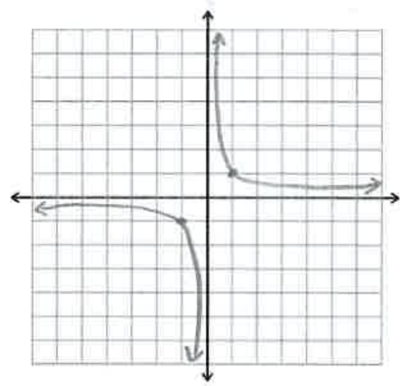
A)



B)



C)



A) Domain $-\infty \leq x \leq \infty$ or $x \in \mathbb{R}$

Range $y \leq 4$

B) Domain $-3 \leq x < 5$

Range $-3 \leq y \leq 2$

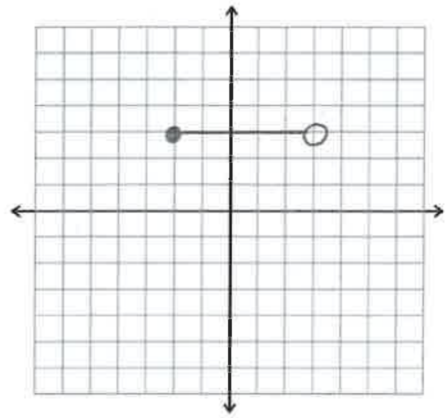
C) Domain $x \in \mathbb{R},\ x \neq 0$

Range $y \in \mathbb{R},\ y \neq 0$

Filled-in vs. Empty points:

When we plot points that exist on the graph, we do so by writing in filled-in dots. But what about points that don't quite exist, like in the line $y = 3,\ where\ -2 \leq x < 3$. For our line, x will be greater than or equal to -2 (filled-in point), but must be less than (and NOT equal to) 3. For this, we would put an empty point (written like a little circle at the point) to represent that we can get infinitely close to 3, but not actually touch it.

Draw the function
$y = 3,\ where\ -2 \le x < 3$



3.2 -Functions

A function is a special type of relation and will be the main focus of our course. Determining whether a relation is a function, is thus a task in and of itself.

Functions

For every x-value there is one, and only one, y-value. In other words: each element in the domain corresponds to exactly one element in the range.

QUICK SUMMARY: Distinct x-values, has a y-value ($1x \rightarrow \#$)!

One-to-one Function (1-1).

A function in which every one value of the domain (x-value) is associated with one value of the range (y-value), and vice-versa.

This means that if f is a 1-1 function, then for each x in the domain of f, there is one, and only one, y in the range, and no y in the range is the image of more than one x in the domain.

QUICK SUMMARY: Distinct x-values go to distinct y-values ($1x \rightarrow 1y$)!

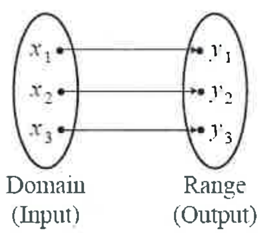
Eg. Take the following three relations:

$y = 2x$	
x	y
0	0
1	2
-2	4
3	6
-4	-8
one-to-one function	

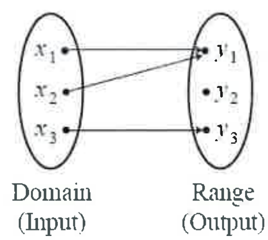
$y = x^2$	
x	y
0	0
3	9
-3	9
5	25
-5	25
Function, but not 1-1	

$y^2 = x$	
x	y
0	0
9	3
9	-3
16	4
16	-4
Not a function, only a relation	

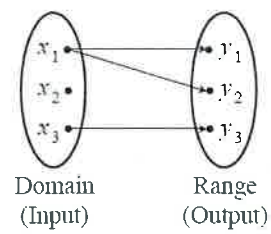
Example:



A one-to-one function



A function, but not one-to-one. Both x_1 and x_2 go to y_1.



Not a function, just a relation. x_1 goes to both y_1 and y_2

Note: The range (output) depends on the domain (input).

<u>Vertical Line Test for Functions</u>

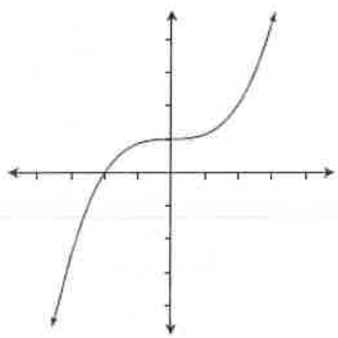
An equation defines y as a function of x if and only if (iff) <u>every vertical line in the coordinate plane intersects the graph of the equation only once</u>. If it intersects more than once than it is not a function.

Horizontal Line Test for One-to-one Functions

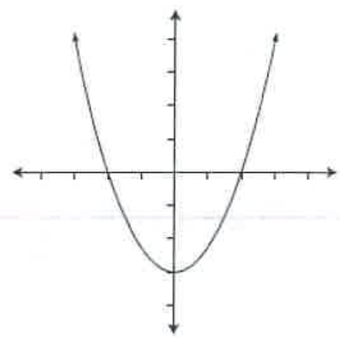
After checking to see if a relation is a function using the vertical line test, you can check to see if it is also one-to-one using the horizontal line test:

A function y is one-to-one function of x iff every horizontal line in the coordinate plane intersects the function at most only once. If it intersects more than once, it is not 1-1.

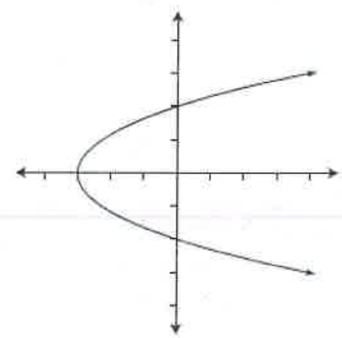
a)



b)



c)



	a)	b)	c)
Function:	Y/N	Y/N	Y/N
One-to-one:	Y/N	Y/N	Y/N

3.3 - Linear Equations

In the previous sections, we learned how to recognize functions from ordered pairs, mapping, and graphs. In this section, we will learn how to recognize functions from algebraic equations.

Expressions are collections of numbers, variables, and operation signs. Often, you will be requested to "simplify" them. They do not have equal signs. (Factoring is an important skill here).

Examples: $10,\ 2y+3,\ 2x^2-4x+5,\ \sqrt{p}$

Equations are mathematical statements that show two expressions are equal. You will often be asked to "solve" them. (What you do to one side, you have to do to the other, is the motto here).

Examples: $y=2,\ y=3x+5,\ y=4(x-3)^2+5$

Linear Equations is any equation of the form $Ax + By = C$ *or* $y = mx + b$! All linear equations are functions, except for one: vertical lines ($x = \#$).
Examples: $y = 4,\ 5x + 3y = 6,\ y = 2x + 3,\ x = 3$

Graphing Linear Equations

One of the main skills you will develop in senior math is your ability to express equations in the form of graphs. There are two main methods to do this: using a table of values (for equations of the form $Ax + By = C$), or slope-intercept form $(y = mx + b)$.

Graphing Linear Equations of the Type $Ax+By=C$

This format is optimally used to solve equations algebraically (and is excellent for quickly finding intercepts). To graph equations of this type, we must create and fill a table of values which we will then plot.
To form a distinct line we must plot three points. Two of these can be the x and y-intercepts, the other(s) can be chosen points that you plug into your original formula.

Steps:

1) Create a table of values, and fill in your findings along the way

2) Find your y-intercept by setting x=0, and solve for y: $(0, y)$.

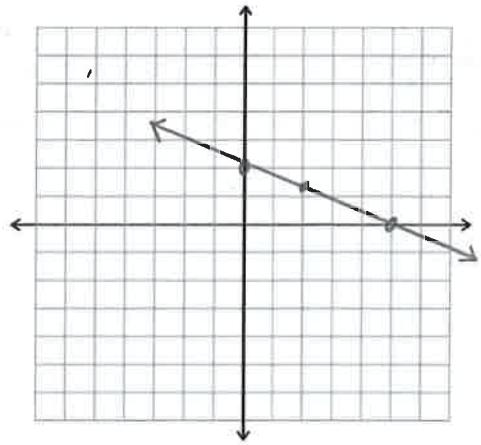
3) Find your x-intercept by setting y=0, and solve for x: $(x, 0)$.

4) Pick and plug-in a random (but easy to use) x-value, and solve for y: (x, y)

5) Once you have your three point-pairs, plot them on the graph!

Example: Graph $2x+5y=10$

$2x+5y=10$	
x	y
0	2
5	0
2	6/5



Solve for the missing values:

$2x+5y=10,\ x=0$
$2(0)+5y=10$
$5y=10$
$y=2$

$2x+5y=10,\ y=0$
$2x+5(0)=10$
$2x=10$
$x=5$

$2x+5y=10,\ x=2$
$2(2)+5y=10$
$4+5y=10$
$5y=6 \Rightarrow y=\frac{6}{5}$

Then plot the points: (0,2) , (5,0) , (2, $\frac{6}{5}$)

Note: Don't forget to use arrows on your line-ends to indicate the line continues indefinitely in both directions.

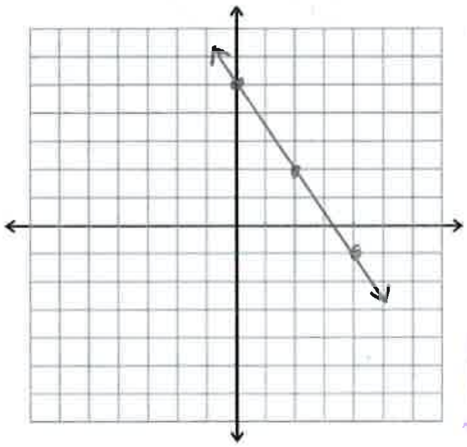
<u>Graphing Linear Equations of the Type $y=mx+b$</u>

This format is optimal for graphing, as we are not only told the y-intercept (b), but also the slope of our line (m).

To solve this type, first plot your b, and then move away from this point based on your m.

Example: $y = -\frac{3}{2}x + 5$

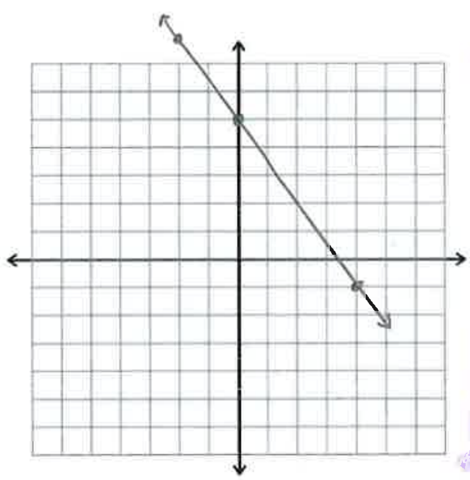
1) Plot b, the y-intercept at $(0,5)$
2) As the slope, $m = -\frac{3}{2}$ this means that we are going down 3, right 2 (or up 3, left 2). Plot the transition on two more points.



You can always solve problems using a table of values to confirm your results. To do this:

1) Select three values of x that are divisible by the denominator of the slope
2) Solve for y in each case.
3) Plot the points.

$y = -\frac{3}{2}x + 5$	
x	y
0	5
-2	8
4	-1



$y = -\frac{3}{2}x + 5,\ x = -2$

$y = -\frac{3}{2}(-2) + 5$

$y = \frac{6}{2} + 5$

$y = 3 + 5 = 8$

$y = -\frac{3}{2}x + 5,\ x = 4$

$y = -\frac{3}{2}(4) + 5$

$y = -\frac{12}{2} + 5$

$y = -6 + 5 = -1$

Summary of the Ordered Pair (x,y)

x	y
Domain	Range
Input (you choose)	Output (solved for)
Independent Variable	Dependent Variable

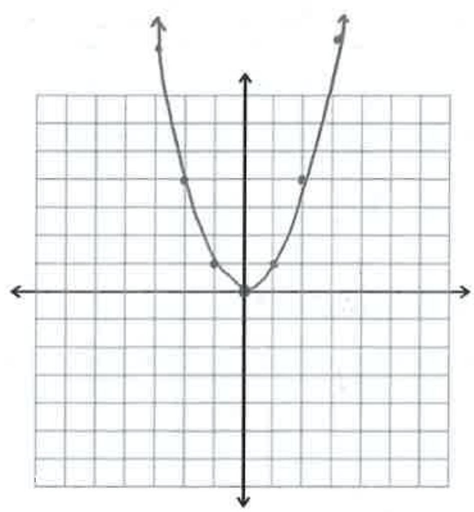
3.4 Non-Linear Equations

The process of graphing non-linear equations is the same as before, but the execution is a little more comprehensive:

1. Use positives, negatives, and zero inputs whenever possible.
2. If any value is to an even power, both the positive and negative values must be used.
3. Use values between 0 and 1 when the variable is in the denominator or is in the exponent.
4. Use as many points as you deem necessary to guarantee the shape of your graph.

Example: Graph $y = x^2$, then determine if it is a function.

$y = x^2$		
x	$y = x^2$	y
0	$y = 0^2$	*0*
1	$y = (1)^2$	*1*
-1	$y = (-1)^2$	*1*
2	$y = (2)^2$	*4*
-2	$y = (-2)^2$	*4*
3	$y = (3)^2$	*9*
-3	$y = (-3)^2$	*-9*

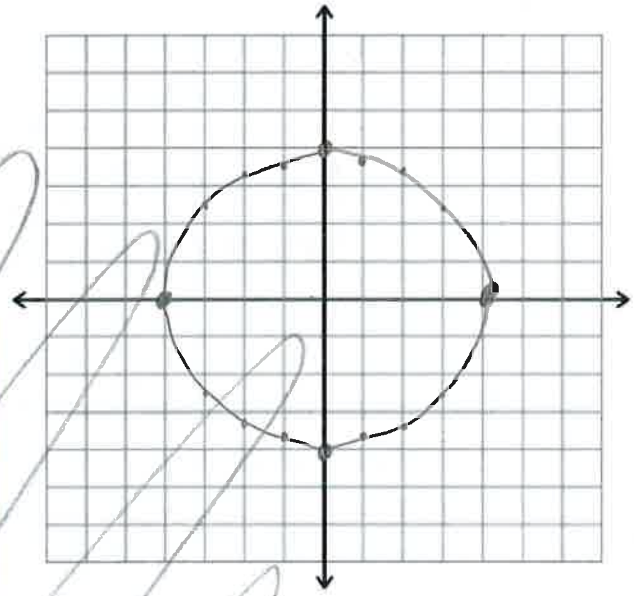


Yes, it is a fuction.
↳ vertical line test
↳ no duplicate x-values

Example: Graph $x^2 + y^2 = 16$

$y = \sqrt{16 - x^2}$

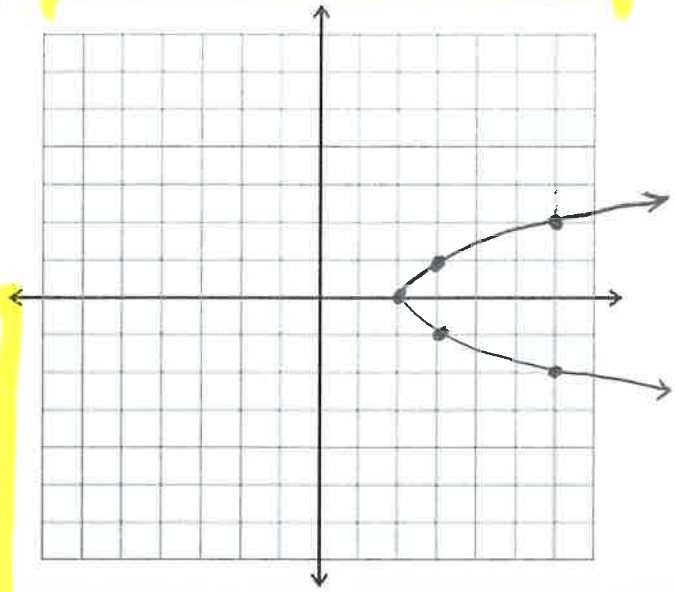
$y = \sqrt{16 - x^2}$		
x	$y = \sqrt{16 - x^2}$	y
0	$y = \sqrt{16 - (0)^2}$	± 4
1	$y = \sqrt{16 - (1)^2}$	$\sqrt{15} \approx \pm 3.87$
-1	$y = \sqrt{16 - (-1)^2}$	$\sqrt{15} \approx \pm 3.87$
2	$y = \sqrt{16 - (2)^2}$	$\sqrt{12} \approx \pm 3.46$
-2	$y = \sqrt{16 - (-2)^2}$	$\sqrt{12} \approx \pm 3.46$
3	$y = \sqrt{16 - (3)^2}$	$\sqrt{7} \approx \pm 2.65$
-3	$y = \sqrt{16 - (-3)^2}$	$\sqrt{7} \approx \pm 2.65$
4	$y = \sqrt{16 - (4)^2}$	0
-4	$y = \sqrt{16 - (-4)^2}$	0
5	$y = \sqrt{16 - (5)^2}$	$\emptyset$
-5	$y = \sqrt{16 - (-5)^2}$	$\emptyset$



Example: Graph $x = y^2 + 2$ then determine if it is a function.

$x = y^2 + 2$		
x	$x = y^2 + 2$	y
2	$x = (0)^2 + 2$ $= 2$	0
3	$x = (1)^2 + 2$ $1 + 2 = 3$	1
3	$x = (-1)^2 + 2$ $1 + 2 = 3$	-1
6	$x = (2)^2 + 2$ $4 + 2 = 6$	2
6	$x = (-2)^2 + 2$ $4 + 2 = 6$	-2
11	$x = (3)^2 + 2$ $= 9 + 2 = 11$	3
11	$x = (-3)^2 + 2$ $= 9 + 2 = 11$	-3
* 0	$0 = y^2 + 2$ $-2 = y^2$ $\sqrt{-2} = \phi$	ϕ
* 1	$1 = y^2 + 2$ $-1 = y^2$ $\sqrt{-1} = y = \phi$	ϕ

} No plotted points

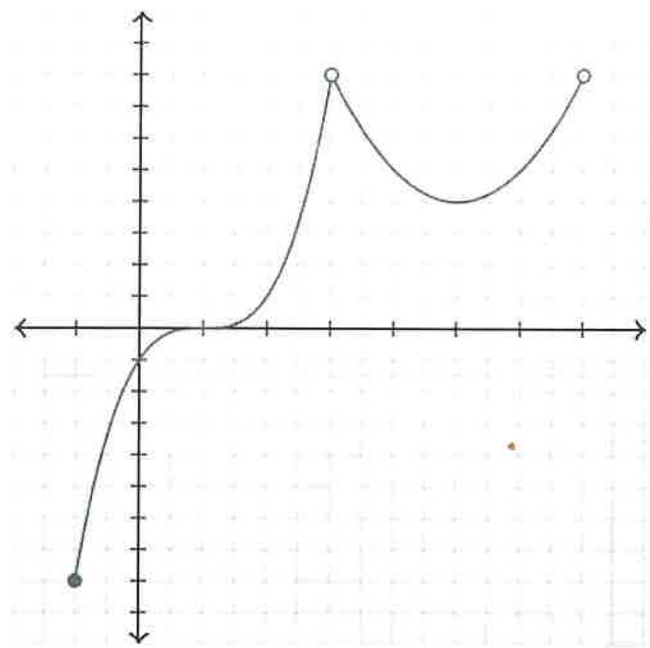


No, it is not a function
↳ vertical line test fails
↳ has duplicate x-values

Appendix

There are several ways of writing the domain and range of a function. You should be able to interpret or explicate the domain and range using any of the following methods:

1) As an inequality
2) As ordered pairs
3) In set notation



1) Inequalities:

Domain: $-2 \leq x$
$x < 14$

Range: $-8 \leq y$
$y < 8$

2) Ordered Pairs

Domain: $[2, 14)$

Range: $[-8, 8)$

3) Set Notation

Domain: $\{x \mid -2 \leq x < 14\}$

Range: $\{y \mid -8 \leq y < 8\}$

Notes: