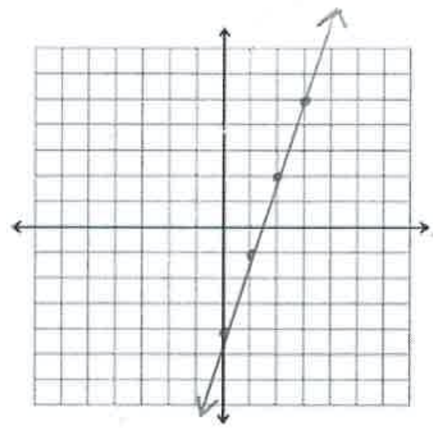
Chapter 6 - Solving Linear Systems

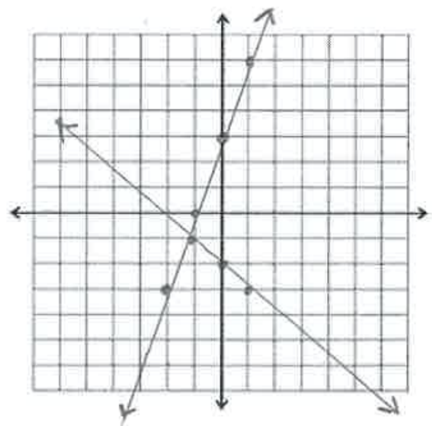
6.1 - Solving Linear Systems by Graphing

So far we have only solved one equation at a time. We are now asked to solve multiple equations and make links between them, which is known as "solving a *system* of equations." We will be exclusively discussing linear functions in our course (linear equations), hence our unit is named: Solving Linear Systems. When graphing two lines, there are only three possible outcomes:

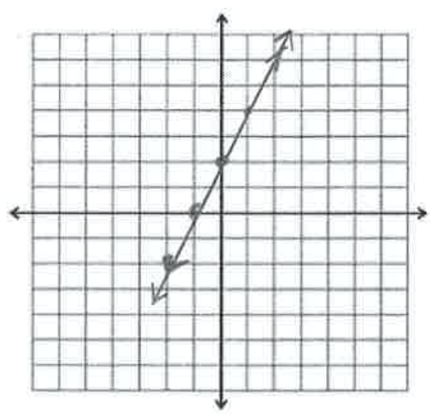
1) That the lines are parallel but have different y-intercepts. They will never intersect each another. These lines have no solution between them we call this an *inconsistent* system with *independent equations*.
   (*i*) $y=3x+2$
   (*ii*) $y=3x-4$



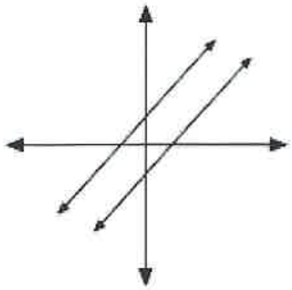
2) The lines are not parallel. The lines will intersect (cross) each other one time. These lines have one solution (the location at which they cross each other), we call this a *consistent system of independent equations*.
   (*i*) $y=-x-2$
   (*ii*) $y=3x+3$



3) The lines are parallel and have the same y-intercept. The lines will intersect each other an infinite number of times...because they are basically the same line. These lines have an infinite number of solutions to the linear equations, we call it a *consistent system of dependent equations*.
   (*i*) $2y=4x+4$ $\Rightarrow$ $y=2x+2$
   (*ii*) $y=2x+2$

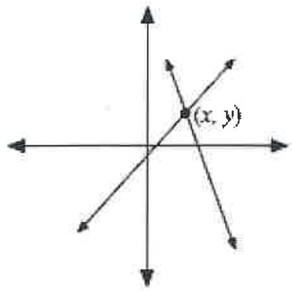


Parallel Lines



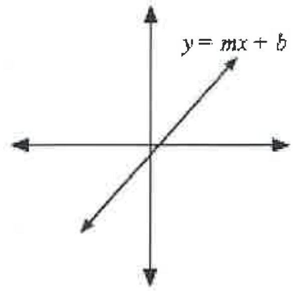
No points in common.
Solution: ∅

Intersecting Lines



One point in common.
Solution: (x, y)

Coincident Lines



Infinitely many points in common.
Solution: $\{(x, y): y = mx + b\}$

<u>To solve linear systems by graphing:</u>

1) Rework the provided equations so they are in a form you can graph (probably $y=mx+b$)
2) Graph both lines.
3) Plug in any points of intersection into both equations. You can verify you have the correct answer if both equations are made true by the particular intersection points.

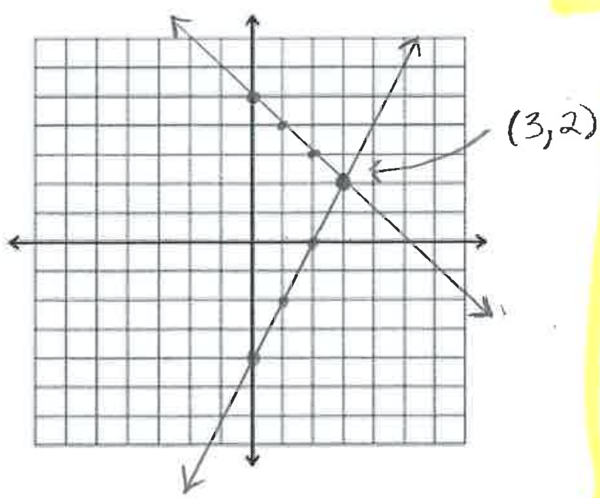
Examples: Solve the systems by graphing. Label any solutions.

1) $x+y=5$ & $y=2x-4$

$\downarrow$

$y=-x+5$

intersection at (3,2)



(1) $x+y=5$

$3+2=5$

$5=5$ ✓

(2) $y=2x-4$

$2=2(3)-4$

$2=6-4$

$2=2$ ✓

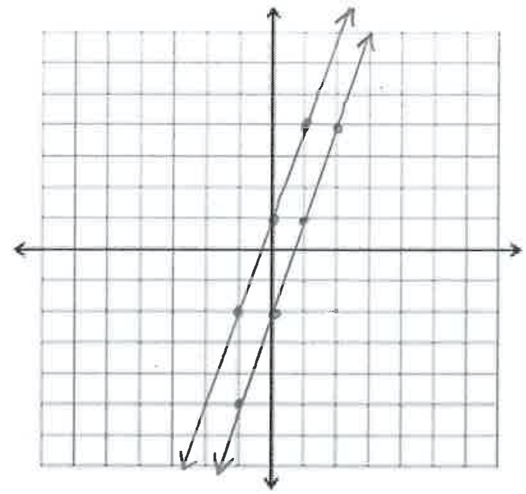
True for both

2) $y=3x+1$ & $-3x+y+2=0$

$\downarrow$

$y=3x-2$

No solution as they never meet.



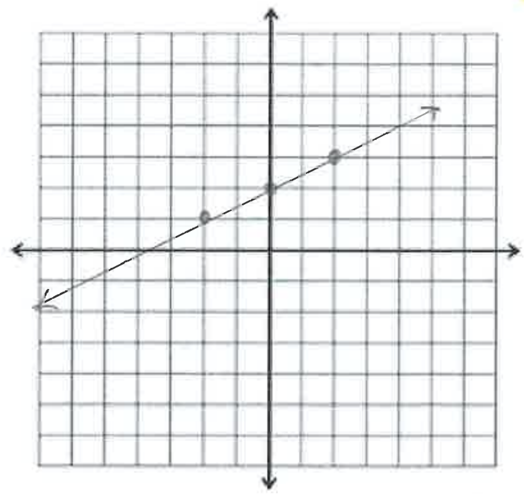
3) $x+2y-4=0$ & $2x+4y=8$

$2y=-x+4$ $\qquad$ $4y=-2x+8$

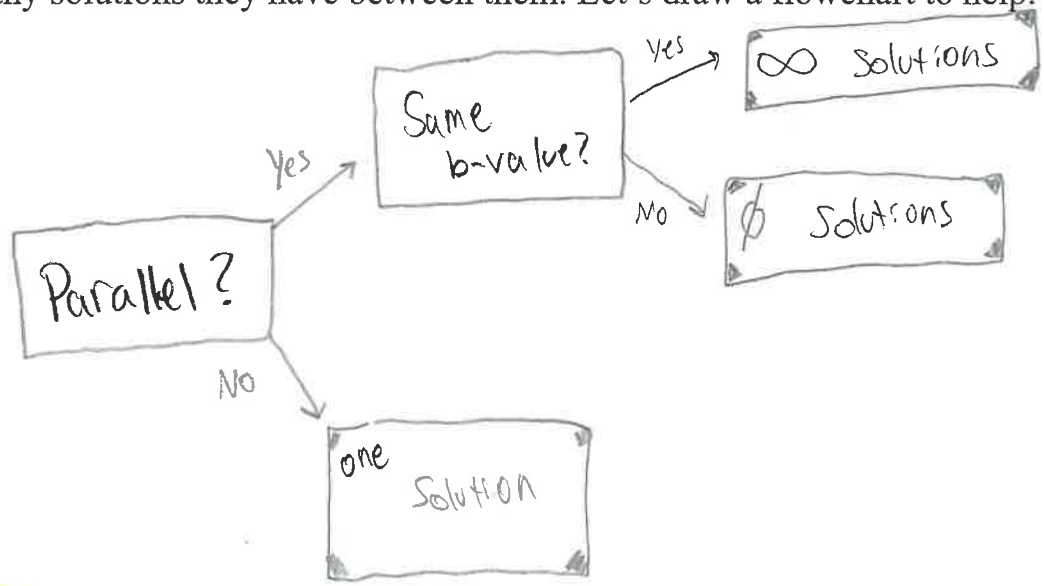
$y=-\frac{1}{2}x+2$ $\qquad$ $y=\frac{-2}{4}x+\frac{8}{4}$

$y=-\frac{1}{2}x+2$

Same line! So ∞ solutions



Keep in mind that you don't necessarily have to graph these functions to know how many solutions they have between them. Let's draw a flowchart to help:



Additionally, you can check to see if an individual point solves the system by plugging in the appropriate points and getting a tautology:
Determine whether the ordered pairs are solutions to the linear system.

Linear System: x-y=-13 & $3x$+y=9
i) (-1, 12)

$x-y=-13$
$(-1)-(12)=-13$
$-13=-13$ ✓

$3(x)+y=9$
$3(-1)+(12)=9$
$-3+12=9$
$9=9$ ✓

Yes, it solves both equations i.e. it solves the system!

ii) (2,3)

$x-y=-13$
$(2)-(3)=-13$
$-1=-13$
X

$3(x)+y=9$
$3(2)+3=9$
$6+3=9$
$9=9$ ✓

No, because it does not solve both equations.

6.2 - Solving Linear Systems by Addition

Solving systems by graphing is an excellent visual and is often used when demonstrating information, but it has its limitations. Solving by graphing is only as good as the accuracy of your graph, which can be poor if we use big values or our answers are fractions. For these reasons, we must also know how to solve systems algebraically - which will be the main focus of the duration of our unit.

One of the methods we will first introduce is the principle of "solving by addition" which utilizes intricacies of the additional property to reduce the number of variables per equation.

GOLDEN RULE FOR THIS UNIT: 1 Equation. 1 Unknown.
You can only solve one variable with one equation! If you have more than one variable (x & y) you must also have the same number of equations. If you have two unknowns you will need two equations to solve it. If you have three unknowns (x, y, & z) you will need three equations.

Take this example: $x+y=10$. There is an infinite number of solutions to this! (1+9, or 2+8 or 1000+(-990) or 9.9+.1, etc...) It is only once we add in another equation $x-y=2$ that it narrows are answers down to one possible solution as both equations must be satisfied for the system to be true: (6,4).

Solving a Linear System by the Addition Method

1) Write both equations of the system in standard form ($Ax+By=C$)
2) Stack the equations just like you would in a regular addition problem, making sure your place values are in the current columns.
3) Multiply one or both of the equations by a constant such that the coefficients of x or y have the same magnitude but differ by their sign.
4) Add the equations, and solve the resulting equation (solve completely for either x or y)
5) Substitute your new found solution back into your original equation to solve for the other variable.
6) Check that both of your new answers actually solve the system (must work for both equations)

Examples: Solve by the addition method.

a)

$$\begin{cases} 3x + 2y = 4 & (1) \\ 2x - 2y = 1 & (2) \end{cases}$$

$$\begin{array}{r} 3x + 2y = 4 \\ + \; 2x - 2y = 1 \\ \hline 5x \quad = 5 \\ \boxed{x = 1} \end{array}$$

$3(1) + 2y = 4$

$3 + 2y = 4$

$2y = 1$

$\boxed{y = \frac{1}{2}}$

Check:

$2(1) - 2(\frac{1}{2}) = 1$

$2 - 1 = 1$

$1 = 1$ ✓

b)

$$\begin{cases} -x - 2y = -3 & (1) \\ 4y + 3x = 2 & (2) \end{cases}$$

$-x - 2y = -3 \quad \times 2$

$3x + 4y = 2$

$$= \begin{array}{r} -2x - 4y = -6 \\ + \; 3x + 4y = 2 \\ \hline x \quad = -4 \end{array}$$

$\boxed{x = -4}$

(1) $-(-4) - 2y = -3$

$4 - 2y = -3$

$7 - 2y = 0$

$7 = 2y$

$\boxed{\frac{7}{2} = y}$

Check: (2) $4y + 3x = 2$

$4(7/2) + 3(-4) = 2$

$14 + -12 = 2$

$\boxed{2 = 2}$ ✓

c)

$$\begin{cases} 5x - 2y = 6 \\ 2x + 3y = 10 \end{cases}$$

$5x - 2y = 6 \quad \times 3$

$2x + 3y = 10 \quad \times 2$

$15x - 6y = 18$

$4x + 6y = 20$

$19x = 38$

$\boxed{x = 2}$

(1) $5(2) - 2y = 6$

$10 - 2y = 6$

$-2y = -4$

$\boxed{y = +2}$

(2) $2(2) + 3(2) = 10$

$4 + 6 = 10$

$\boxed{10 = 10}$ ✓

d)

$$\begin{cases} \frac{x}{2} - \frac{y}{3} = \frac{7}{12} & \times 12 = 6x - 4y = 7 \quad \times 2 = 12x - 8y = 14 \\ \frac{x}{8} + \frac{y}{9} = 0 & \times 72 = 9x + 8y = 0 \end{cases}$$

$+ \; 9x + 8y = 0$

$21x = 14$

$x = \frac{14}{21} = \boxed{\frac{2}{3}}$

(1) $6\left(\frac{2}{3}\right) - 4y = 7$

$4 - 4y = 7$

$-4y = 3$

$\boxed{y = -\frac{3}{4}}$

Check:

(2) $9\left(\frac{2}{3}\right) + 8\left(\frac{-3}{4}\right) = 0$

$6 - 6 = 0$

$\boxed{0 = 0}$ ✓

6.3 - Solving Linear Equations by Substitution

The addition method learned in 6.2 is very useful for equations where there is evident overlap. In 6.3 we will learn arguably the most important skill of our course: solving systems by substitution. This method takes advantage of the definition of equality and allows you to reduce the number of variables per equation by replacing once variable with its relation to the other.

<u>Method for Solving Linear Systems by the Substituon</u>

1. Label your equations in the system (1) and (2)
2. For equation (1), move the equation around so you solve for a particular variable. In essence, convert your equation from standard (Ax+By=C) to slope-intercept form (y=mx+b).
3. Label this reworked equation (3).
4. In (3) your equation now has one variable (y) equaling a function (mx+b). Take the function part (mx+b) and plug it in for the y in equation (2) (This is the "substitution" part). Now equation (2) only has one type of variable (in our case x) in it.
5. Solve (2) for x.
6. Now that x has been solved for, you can plug in its value back into (3) and solve for your other variable.
7. Record your two found answers as ordered pairs.

Examples: Solve by the substitution method.

a)

$$\begin{array}{l} (1) \\ (2) \end{array} \begin{cases} -5x+2y=9 \\ y=7x \end{cases}$$

$-5x + 2(7x) = 9$

$-5x + 14x = 9$

$9x = 9$

$\boxed{x = 1}$

$\boxed{1, 7}$

$y = 7(1)$

$\boxed{y = 7}$

check:

$-5(1) + 2(7) = 9$

$-5 + 14 = 9$

$\boxed{9 = 9}$ ✓

b)

$$\begin{cases} y = 6x - 11 & (1) \\ -2x - 3y = -7 & (2) \end{cases}$$

$-2x - 3(6x - 11) = -7$

$-2x - 18x + 33 = -7$

$-20x + 33 = -7$

$-20x = -40$

$\boxed{x = 2}$

$y = 6(2) - 11$

$\boxed{y = 1}$

Check:

(2) $-2(2) - 3(1) = -7$

$-4 - 3 = -7$

$\boxed{-7 = -7}$ ✓

$\boxed{2, 1}$

c)

$$\begin{cases} -7x - 2y = -13 & (1) \\ x - 2y = 11 & (2) \end{cases}$$

$\longrightarrow -2y = -x + 11 \Rightarrow y = \frac{1}{2}x - \frac{11}{2}$ (3)

(1) $-7x - 2\left(\frac{1}{2}x - \frac{11}{2}\right) = -13$

$-7x - x + 11 = -13$

$-8x = -24$

$\boxed{x = 3}$

(3) $y = \frac{1}{2}(3) - \frac{11}{2}$

$y = \frac{3}{2} - \frac{11}{2} = \frac{-8}{2} = \boxed{-4 = y}$

Check:

(2) $3 - 2(-4) = 11$

$3 + 8 = 11$

$\boxed{11 = 11}$ ✓

$\boxed{3, -4}$

d)

(1) $-5x+y=-3 \Rightarrow y=5x-3$ (3)

(2) $3x-8y=24$

(3) → (2)

$3x-8(5x-3)=24$

$3x-40x+24=24$

$-37x=0$

$\boxed{x=0}$

x → (3)

$y=5(0)-3$

$\boxed{y=-3}$

x, y → (2)

$3(0)-8(-3)=24$

$\boxed{0+24=24}$ ✓

$\boxed{(0,-3)}$

e)

(1) $-3x+3y=4$

(2) $-x+y=3 \longrightarrow y=x+3$ (3)

(3) → (1)

$-3x+3(x+3)=4$

$-3x+3x+9=4$

$9=4$??

∴ No Solution

$\boxed{y=x+3}$

&

$3y=3x+4 \Rightarrow \boxed{y=x+\frac{4}{3}}$

They are parallel, but have different y-int.

∴ No Solutions!

6.4 - Problem Solving with Two Variables

While this section often causes the most problems, it is merely an extension of the same concepts that have already been covered. Before we do some examples, let's go over some strategies for problem-solving.

1) Remember that you will be creating formulas with two variables, and thus two equations.
2) Determine exactly what is being asked of you; set your variables so they actually solve for the wanted information.
3) Form a transition statement (that incorporates both written English and math notation) before attempting to finalize a formula
4) When in doubt, write it out! If you feel you have a direction (or don't) write anything you can to try and spark something.

Examples:

a) The difference of two numbers is 3. Their sum is 13. Find the numbers.

let x = the first #
let y = the second #

(1) $x-y=3$ (2) $x+y=13$

(3) $x=3+y$ (3)→(2) $(3+y)+y=13$

$3+2y=13$

$2y=10$

$\boxed{y=5}$

y→(3) $x=3+5$

$\boxed{x=8}$

x & y → (2) $8+5=13$ ✓

$\boxed{(8,5)}$

b) Flying to Toronto with a tailwind a plane averaged 158 km/h. On the return trip, the plane only averaged 112 km/h while flying back into the same wind. Find the speed of the wind and the speed of the plane in still air.

let p = speed of plane in still air

let w = speed of wind

(1) $p+w=158$

(2) $p-w=112$

$$2p = 270$$

$$\frac{2p}{2} = \frac{270}{2}$$

$\boxed{p=135}$

$p+w=158$

$135+w=158$

$-135 \quad -135$

$\boxed{w=23}$

135 km/h = Plane
23 km/h = wind

c) Our school is selling tickets to the seasons band concert. Staff tickets cost $5 and student tickets cost $4. On the day of the concert, you look for the ticket spreadsheet but seem to have lost it! Luckily you know that you sold 240 tickets and made exactly $1000. How many tickets were sold to staff and how many were sold to students?

Let f = # staff tickets sold

Let n = # student tickets sold

1) $f+n=240$

2) $5f+4n=1000$

$n=240-f$

$5f+4(240-f)=1000$

$5f+960-4f=1000$

$-960 \quad -960$

$\boxed{f=40}$

$f+n=240$

$40+n=240$

$-40 \quad -40$

$\boxed{n=200}$

200 student tickets
40 staff tickets

6.5 - Arithmetic Sequences

A sequence is simply a list of numbers in a pattern. Each number in the sequences, based on its location, is known as a term.

Eg A. {2, 4, 6, 8, 10, 12} Is a sequence with six terms (term #1 is 2, term #4 is 8, the last term is 12). Because it ends, it is said to be a finite sequence.

Eg B. {4, 7, 10, 13, 16, 19…} Is a sequence that never ends, so it is known as an infinite sequence. Term #1 is now 4, and term #4 is 13, but there is no last term in our sequence - hence it being called an infinite sequence.

As you can see, a sequence is like a list of numbers - the initial term is called the "first term," the next term is called the "second term" and thus the terms are elements of the natural numbers.

A sequence is written using subscript notation.
For example: $t_1, t_2, t_3, t_4, \ldots t_n$
The subscript identifies the term of the sequence, so for our previous examples:

Sequence A: $t_1 = 2\ t_3 = 6,\ t_6 = 12$
Sequence B: $t_1 = 4\ t_2 = 7\ t_5 = 16\ t_8 = 25\ t_{100} = 301$

What are the formulas for Sequence A and B?

A: {2, 4, 6, 8, 10, 12}
$t_n = 2n$

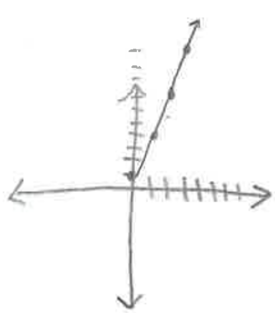
x	x	n	1	2	3	4	5	6
y	$f(x)$	t_n	2	4	6	8	10	12

n	t_n
$1 \to$	2
$2 \to$	4
$3 \to$	6

$t_n = 2n$

$y = mx + b \leftarrow$ y-int
$f(x) = mx + b \swarrow$
$t_n = mn + b \swarrow$
↑
Slope $= \dfrac{\text{rise } 3}{\text{run } 1}$

B: {4, 7, 10, 13, 16, 19…}
$t_n = 3n + 1$



x	x	n	1	2	3	4	5	6
y	$f(x)$	t_n	4	7	10	13	16	19

1
↑ b

What are the first four terms of the following sequences:

i) $a_n = \frac{n+2}{n}$

$a_1 = \frac{1+2}{1} = \frac{3}{1} = \boxed{3}$ $\qquad a_2 = \frac{2+2}{2} = \frac{4}{2} = \boxed{2}$ $\qquad \left\{3, 2, \frac{5}{3}, \frac{3}{2}, \dots\right\}$

$a_3 = \frac{3+2}{3} = \boxed{\frac{5}{3}}$ $\qquad a_4 = \frac{4+2}{4} = \frac{6}{4} = \boxed{\frac{3}{2}}$

ii) $b_n = (n-1)^n$

$b_1 = (1-1)^1 = 0^1 = \boxed{0}$ $\qquad b_2 = (2-1)^2 = 1^2 = \boxed{1}$

$b_3 = (3-1)^3 = 2^3 = \boxed{8}$ $\qquad b_4 = (4-1)^4 = 3^4 = 81$

$\boxed{\{0, 1, 8, 81\}}$

iii) $t_n = 2n - 11$

$t_1 = 2(1) - 11 = 2 - 11 = \boxed{-9}$ $\qquad t_2 = 2(2) - 11 = 4 - 11 = \boxed{-7}$

$t_3 = 2(3) - 11 = 6 - 11 = \boxed{-5}$ $\qquad t_4 = 2(4) - 11 = 8 - 11 = \boxed{-3}$

$\{-9, -7, -5, -3 \dots\}$

There is another way of defining a sequence, known as a recursive sequence. A recursive sequence can only really be built one unit at a time as it requires the value from the previous term to be solved (notice that we are not asked for n, but rather n-1 in our formula).

For these to work, we must know the value of the first term, as well as the recursive formula

What are the first four terms of this recursive sequence

$a_1 = 5,\ a_n = 2(a_{n-1}) + 1$

$\boxed{a_1 = 5}$

$a_2 = 2(a_{2-1}) + 1$
$a_2 = 2(a_1) + 1$
$a_2 = 2(5) + 1$
$\boxed{a_2 = 11}$

$a_3 = 2(a_{3-1}) + 1$
$a_3 = 2(a_2) + 1$
$a_3 = 2(11) + 1$
$\boxed{a_3 = 23}$

$a_4 = 2(a_{4-1}) + 1$
$a_4 = 2(a_3) + 1$
$a_4 = 2(23) + 1$
$a_4 = 46 + 1$
$\boxed{a_4 = 47}$

$\{5, 11, 23, 47\}$

Sigma Notation

Sometimes you will be asked to find the sum of all the terms in a sequence (this is known as an "Arithmetic Series" which we will discuss at length in the next section).

Writing a series out with our expanded notation can be long and tedious, so mathematicians created new notation - known as Sigma Notation (using the Greek letter sigma (Σ)) - that condenses our series into a single compact expression.

OLD: $\{t_n\} = a_1 + a_2 + a_3 + \ldots + a_n$ => NEW (SIGMA): $\sum_{k=1}^{n} a_k$

We read this new form from bottom to top:

Bottom: k=*1* is the called the index of the sum, and it shows where the summation starts (often at 1), but you may see k=0 if that is where we want to start our series.
Top: n represents when the summation ends (the last number you will plug-in)
Right: a_k is the function that will determine the rate of your sequence.

Example of how Sigma condenses our problem:

OLD: $\{t_{85}\} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 \ldots + 85$ (can't even write the whole thing out, it would take too long)

NEW (SIGMA): $\sum_{k=1}^{85} (k)$

In general, the summation $\boxed{\sum_{k=1}^{n} a_k \text{ has } n\text{-}k\text{+}1}$ total terms. You can see in our above example, the series has 85 terms: n=85 and k=1 and 85-1+1 = 85.

Find the sum of the following sequences:

i) $\sum_{k=1}^{4} 2k-1$

$a_1)\ 2(1)-1 = 2-1 = \boxed{1}$

$a_2)\ 2(2)-1 = 4-1 = \boxed{3}$

$a_3)\ 2(3)-1 = 6-1 = \boxed{5}$

$a_4)\ 2(4)-1 = 8-1 = \boxed{7}$

$1+3+5+7 = \boxed{16}$

ii) $\sum_{k=0}^{3} k(k+2)$

$a_0 = 0(0+2) = 0$

$a_1 = 1(1+2) = 1(3) = 3$

$a_2 = 2(2+2) = 2(4) = 8$

$a_3 = 3(3+2) = 3(5) = 15$

$0+3+8+15$

$= \boxed{26}$

Write the provided sum using sigma notation:

1) 3+5+7+9+11+13+15+17

$\sum_{k=1}^{8}(2k+1)$

2) 100 + 50+ 25 …

$$\sum_{k=1}^{\infty} 100\left(\tfrac{1}{2}\right)^{k-1}$$

3) $0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

$$\sum_{k=0}^{4}\left(\frac{k}{k+1}\right)$$

As you can see, finding the formulas for such sequences can be very challenging. Fortunately, there is a new formula that we can apply to make this much easier.

Along with the starting and ending point of a sequence, the difference between each term is vitally important. When the difference between successive terms is always the same number (like in the sequence 2, 8, 14, 20, 26) the sequence is said to be arithmetic, with a common difference of 6 ($d = +6$).

What is the difference in this sequence?
{24, 19, 14, 9…}

$d = -5$

Let's derive our arithmetic sequence formula by observing what happens when we expand the first few terms of a hypothetical sequence.

1st term: $a_1 = a_1$
2nd term: $a_2 = a_1 + d$
3rd term: $a_3 = a_2 + d \; OR \; a_1 + d + d \; = \; a_1 + 2d$
4th term: $a_4 = a_3 + d \; OR \; a_1 + d + d + d \; = \; a_1 + 3d$
Notice: The number of differences (jumps) is always one less than the number of terms (positions)

General Formula: $t_n = a + (n - 1)d$

*Fine print: For an arithmetic sequence $\{t_n\}$ whose first term is a, with a common difference of d and n is a natural number.

Examples:

i) Find the 8th number in the arithmetic sequence using the traditional approach as well as our new formula. Also, find the 853rd term.

$y = 5x - 2$

{3, 8, 13, 18…}

0 1 2 3 4 5 6 7 8

Traditional: -2 3 8 13 18 23 28 33 $\boxed{38}$

Formula: $t_n = a + (n-1)d$ $a = 3$ $n = 8$ $d = 5$

$t_8 = 3 + (8-1)5$ $t_{853} = 3 + (853-1)5$

$t_8 = 3 + (7)5$ $t_{853} = 3 + (852)(5)$

$t_8 = 3 + 35$ $t_{853} = 3 + 4260$

$\boxed{t_8 = 38}$ $\boxed{t_{853} = 4263}$

ii) Which term in the arithmetic sequence 11, 15, 19, … has a value of 367?

x : n $\quad$ y : t_n

$t_n = a + (n-1)d$ $\quad$ $a = 11$ $\quad$ $n = ?$

$d = 4$ $\quad$ $t_n = 367$

$367 = 11 + (n-1)4$
$-11 \quad -11$

$\frac{356}{4} = \frac{(n-1)4}{4}$ $\quad$ $89 = n - 1$
$+1 \quad +1$

$\boxed{n = 90}$

iii) The 9th term of an arithmetic sequence is 72, and the 20th term is 39. Find the first term.

$a = ?$

OR

$t_9 = 72$ $\quad$ $72 = a + (9-1)d$ $\quad$ $72 = a + 8d$

$t_{20} = 39$ $\quad$ $39 = a + (20-1)d$ $\quad$ $39 = a + 19d \quad \times -1$

$\frac{39-72}{20-9} = \frac{-33}{11} = \boxed{-3}$

$72 = a + 8d$
$+ \; -39 = -a - 19d$

$\frac{33}{-11} = \frac{-11d}{-11}$ $\quad$ $\boxed{d = -3}$

$39 = a + (20-1)(-3)$
$39 = a + (19)(-3)$
$39 = a + -57$

$\boxed{a = 96}$

iv) Find x so that 3x+2, 2x-3, and 2-4x are consecutive terms of an arithmetic sequence.

lets look at 3 consecutive numbers 9, 10, 11

If we take the average of the outside values, we get the inside value! Eg $\frac{9+11}{2} = 10$!

$\therefore \frac{(3x+2)+(2-4x)}{2} = 2x - 3$

$\frac{3x+2+2-4x}{2} = 2x - 3$

$\frac{-x+4}{2} = 2x - 3$

$(-x+4) = 2(2x-3)$

$-x + 4 = 4x - 6$

$10 = 5x$

$\boxed{x = 2}$ ← answer

Check:

1) $3(2)+2 = 8$
2) $2(2)-3 = 1$
3) $2-4(2) = -6$

$\boxed{d = -7}$

6.6 - Arithmetic Series

In 6.5 we learned about addition patterns, known as an arithmetic sequence; in 6.6 we will officially discuss when these patterns are continually added (summed) together, known as an arithmetic series.

Deriving the Sum Formula for Finite Arithmetic Series

If $a_1, a_2, a_3, \ldots, a_n$ is a finite arithmetic sequence, then $a_1 + a_2 + a_3 + \cdots + a_n$ is a finite arithmetic series.

Let $d =$ the common difference, $S_n =$ the sum of the series.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$
$$= a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + (n-1)d) \quad \text{(equation 1)}$$

Let $l = a + (n-1)d$ *(the last term)*

Writing the sum in reverse order: $S_n = l + (l - d) + (l - 2d) + \cdots + a$ (equation 2)

Adding equations 1 and 2: $2S_n = (a + l) + (a + d + l - d) + (a + 2d + l - 2d) + \cdots + (a + l)$
$= (a + l) + (a + l) + (a + l) + \cdots + (a + l)$

But $(a + l)$ appears n times. Therefore $2S_n = n(a + l) \rightarrow S_n = \frac{n}{2}(a + l)$.

Also, $S_n = \frac{n}{2}(a + l) \rightarrow S_n = \frac{n}{2}(a + a + (n-1)d) \rightarrow S_n = \frac{n}{2}(2a + (n-1)d)$.

Sum of an Arithmetic Series Formula

The sum of the first n terms of an arithmetic series (S_n) is given by:

$S_n = \frac{n}{2}(a + l)$ which can also be written as $S_n = \frac{n}{2}(2a + (n-1)d)$

Where $a =$ the first term, $l =$ the last term, and $d =$ the common difference.

Use if you <u>know</u> the last term

Use if you <u>do not</u> know the last term

Example 1) Find the sum of the positive integers from 1 to 10 inclusive using a traditional method, as well as our formula.

Traditional: $1+2+3+4+5+6+7+8+9+10=\boxed{55}$

formula: $S_{10}=\frac{n}{2}(a+l) \Rightarrow S_{10}=\frac{10}{2}(1+10) \Rightarrow$

$= 5(11)$

$\boxed{S_{10}=55}$

Eg 2) Find the sum of the first 35 terms of the series 12+18+24…

$S_n=\frac{n}{2}(2a+(n-1)d)$

$S_{35}=\frac{35}{2}(2(12)+(35-1)6)$

$S_{35}=\frac{35}{2}(24+34(6))$

$S_{35}=\frac{35}{2}(24+204)$

$S_{35}=\frac{35}{2}(228)$

$\boxed{S_{35}=3990}$

Eg 3) Find the sum of the series 8+11+14+...+101.

① Use sequence formula to find out which term 101 is.

$t_n=a+(n-1)d$

$101=8+(n-1)3$

$93=(n-1)3$

$31=n-1$

$\boxed{n=32}$

$S_n=\frac{n}{2}(a+l)$

$S_{32}=\frac{32}{2}(8+101)$

$S_{32}=16(109)$

$\boxed{S_{32}=1744}$

Eg 4) Evaluate $\sum_{k=1}^{200}(3k-2)$ → Starts at 1, Ends at 200

Difference?

① $S_n = \frac{n}{2}(2a+(n-1)d)$

$S_{200} = \frac{200}{2}(2(1)+(200-1)3)$

$S_{200} = 100(2+199(3)) = \boxed{59900}$

OR

② $S_n = \frac{n}{2}(a+l) \Rightarrow \frac{200}{2}(1+598) = 100(599) = \boxed{59900}$

$t_1 = (3(1)-2)$; $t_2 = (3(2)-2)$

$t_1 = 3-2$; $t_2 = 6-2$

$t_1 = 1 \leftarrow a$; $t_2 = 4$ ∴ $\boxed{d=3}$

$t_{200} = (3(200)-2) = 600-2 = 598$

Eg 5) Write 6+10+14+ … +138 in summation (sigma) notation.

$t_n = a+(n-1)d$

$138 = 6+(n-1)4$

$132 = (n-1)4$

$\frac{132}{4} = n-1$

$33 = n-1$

$\boxed{n=34} \leftarrow$ ends

$d=4$

Starts

$t_1 = 6 = 4(1) + \underline{2}$

$t_k = 4k+2$

$\sum_{k=1}^{34}(4k+2)$

Eg 6) The sum of the first n terms of an arithmetic sequence is $S_n = 4n^2 + 2n$. Find the common difference (d).

$S_n = 4n^2+2n$

first term a_1 → $S_1 = 4(1)^2+2(1)$

$S_1 = 4+2 = \boxed{6}$

not the second term

$S_2 = 4(2)^2+2(2)$

$S_2 = 16+4 = \boxed{20}$

But

$S_2 = a_1 + a_2 = 20$

$6 + a_2 = 20$

second term → $a_2 = 14$

$d = a_2 - a_1$

$d = 14-6$

$\boxed{d=8}$

Check:

$a_1 + a_1 + d$

$6+(6+8) = 20$

$6+14 = 20$

$20=20$ ✓

Eg 7) Find the sum of all multiples of 5 between 101 and 1001.

The first multiple of 5 after 101 is 105
The last multiple of 5 before 1001 is 1000

Find the number of terms: $t_n = a + (n-1)d$

$1000 = 105 + (n-1)5$

$895 = (n-1)5$

$179 = n-1$

$\boxed{n=180}$

$S_{180} = \frac{n}{2}(a+l)$

$S_{180} = \frac{180}{2}(105+1000)$

$S_{180} = 90(1105) = \boxed{99450}$

Eg 8) The sum of three consecutive terms of an arithmetic sequence is 3. The sum of their squares is 75. Find the three numbers.

three consecutive terms of an arithmetic sequence are separated by d's

let a = first term

$(a) + (a+d) + (a+d+d) = 3 \Rightarrow 3a + 3d = 3$

① $a + d = 1 \Rightarrow$

$\hookrightarrow a = 1-d$

$(a)^2 + (a+d)^2 + (a+d+d)^2 = 75$

$(1-d)^2 + (1)^2 + (1+d)^2 = 75$

$(1-d)(1-d) + 1 + (1+d)(1+d) = 75$

$(1-2d+d^2) + (1) + (1+2d+d^2) = 75$

$3 + 2d^2 = 75 \rightarrow 2d^2 = 72$

$d^2 = 36 \rightarrow \boxed{d = \pm 6}$

⊕ $a + 6 = 1$

$\boxed{a=-5}$ $+6 = \boxed{1}$ $+6 = \boxed{7}$

⊖ $a - 6 = 1$

$\boxed{a=+7}$ -6 $\boxed{1}$ -6 $\boxed{-5}$