1.1 – Ratios & Rates

Key

Ratio definition

A **ratio** is a comparison of two quantities, written as a fraction. The ratio **a** to **b** can Be written in the following ways: $\frac{a}{b}$ or **a : b**

Example 1: You eat 6 pieces out of an 8 piece pizza. Express as a ratio in both forms in lowest terms, and then as a decimal.

$$\frac{6 \text{ pieces}}{8 \text{ pieces}} = \frac{6^{\div 2}}{8^{\div 2}} = \frac{3 \text{ pieces}}{4 \text{ pieces}} = \frac{3}{4} \text{ or } 3:4 \qquad \frac{3}{4} = 3 \div 4 = 0.75$$

Since the units in the example above cancel out (both have units of 'pieces of pizza'), they are called **like quantities**.

$$\frac{3 \cancel{\text{pieces}}}{4 \cancel{\text{pieces}}} = \frac{3}{4}$$

Rate definition

When making a ratio of two **unlike quantities** (they have different units), this is called a **rate**.

Example 2: Your pay is \$90 for 8 hours of work. What is your rate of pay? Write answer in lowest terms as well.

$$\frac{\$90}{8 \text{ hrs}} = \frac{90^{\div 2}}{8^{\div 2}} = \frac{\$45}{4 \text{ hrs}}$$

Unit Rate definition

A **unit rate** is a rate in which the two different quantities have a denominator equal to **1.**

Example 3: Write Example 2 as a unit rate. What does this represent?

$$\frac{\$45}{4 \text{ hrs}} = 45 \div 4 = 11.25 = \frac{\$11.25}{1 \text{ hr}} = \$11.25/\text{hr}$$

Example 4: Write each as a unit rate:

a) You are paid \$80 for 4 hours of work

$$\frac{\$80}{4 \text{ hr}} = \frac{\$20}{1 \text{ hr}} = \$20/\text{hr}$$

b) Car A travels 130km on 9L of gasoline and car B travels 220km on 16.4L of gasoline. Which has the better gas mileage?

$$\text{Car A: } \frac{130 \text{ km}}{9 \text{ L}} = 14.\bar{4} \text{ km/L} \qquad \text{Car B: } \frac{220 \text{ km}}{16.4 \text{ L}} = 13.4 \text{ km/L}$$

Car A has the better mileage

Example 5: Convert 35km/hr to miles/hr. 1 mile = 1.609km

$$\frac{35\cancel{km}}{1\,hr} \times \frac{1\,mile}{1.609\cancel{km}} = \frac{35}{1.609} = \frac{21.75\,mi}{1\,hr} = 21.75\,mi/hr$$

Example 6: J.J. runs 26 yards in 3.8 seconds. What is the unit rate?

$$\frac{26\,yd}{3.8\,s} = 6.84\,yd/s$$

What is the unit rate in miles per hour? 1760 yards = 1 mile 1 hr = 3600s

$$\frac{6.84\cancel{yd}}{1\cancel{s}} \times \frac{1\,mile}{1760\cancel{yd}} \times \frac{3600\cancel{s}}{1\,hr} = \frac{6.84 \times 3600}{1760} = 14\,mi/hr$$

Example 7: A woman bicycles 300m in 24seconds. What is her speed in km/hr?

1 km = 1000m
1 hr = 3600s

$$\frac{300\cancel{m}}{24\cancel{s}} \times \frac{1\,km}{1000\cancel{m}} \times \frac{3600\cancel{s}}{1\,hr} = \frac{1\,080\,000\,km}{24000\,hr} = 45\,km/hr$$

Example 8: If a car has a mileage of 32km/L, how many km can the car travel by burning 17L of gas?

Think:
Car can go 32km with 1 L
so how far with 17L ?

Multiply!

$32 \times 17 = 544\,km$

How many litres of fuel are needed for the car to travel 250km?

Think:
Car can go 32km with 1 L
How many litres needed for 500km?

Divide!

$500 \div 32 = 15.625L$

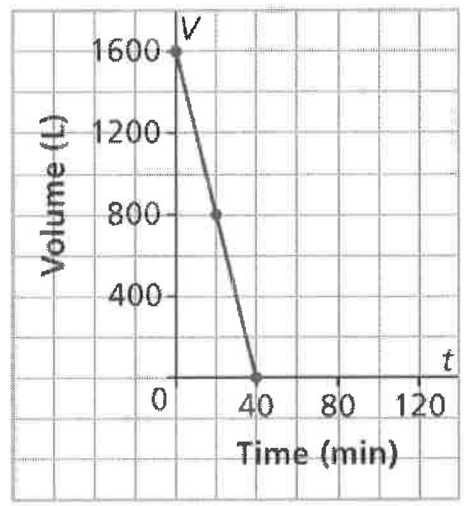
1.2 – Visualizing Rates

Rate of Change

Rate of change is the same thing as **rate**, which is a description of how one thing changes in relation to another.

Last day, we learned how to calculate rate, or **rate of change** mathematically. We can also graph the relationship between two variables. Example 1 is shown below:

Emptying a Hot Tub



Can you write a **rate, or rate of change** for the situation?

draining $\frac{-1600L}{40\,min}$

Can you write a **unit rate**? What does it represent?

$\frac{-1600L}{40min} = -40L/min$ Tub is emptying 40L per minute

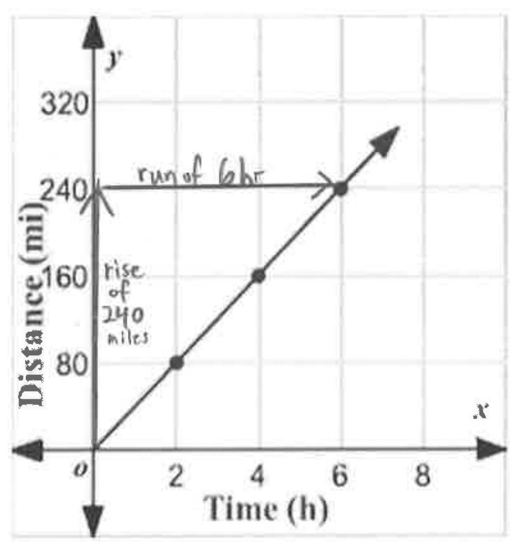
Can you connect the **rate of change** with a key concept learned in Math 10?

slope! The rate of change is the slope of the line!

Therefore, the **rate of change** for a situation is the **slope** of the line when the relationship is graphed.

Because **slope** = $\frac{rise}{run}$, numerator is the unit from the ***y-axis***, and the denominator is the unit from the ***x-axis***.

Example 2: Driving on a country road



a) Calculate the **slope**, or **rate of change**:

$\frac{rise}{run} = \frac{240mi}{6\,hr}$

b) Calculate the **unit rate**. What does this represent?

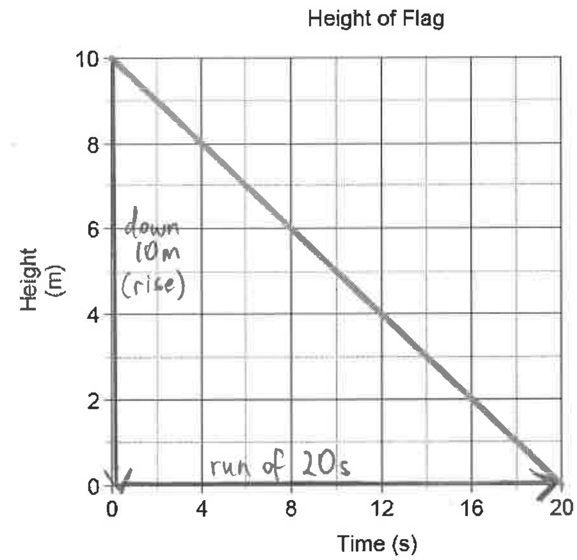
$\frac{240mi}{6\,hr} = 40\,mi/hr$ Represents the speed of the car!

c) Calculate the slope another way:

$slope = \frac{rise}{run} = \frac{80mi}{2\,hr}$ ← up 80 (rise) on graph ← right 2 (run) on graph

$= 40\,mi/hr$

Example 3: Calculate the speed at which the flag is lowered.



$$\text{slope} = \text{speed} = \frac{\text{rise}}{\text{run}} = \frac{-10\text{m}}{20\text{s}}$$

down

$$= -0.5\text{m/s}$$

flag is lowered half a metre per second

Example 4: A taxi charges \$3.00 plus \$1.20 per km. Draw a graph and determine:

a) The cost of travelling 15km

b) The distance travelled for \$27

d	\$
0	3.00
1	4.20
2	5.40
3	6.60
4	7.80
5	9.00
10	15.00
15	21.00
20	27.00

$$\text{slope} = \frac{\text{rate of change}}{} = \frac{\$24}{20\text{km}} = \$1.20/\text{km}$$

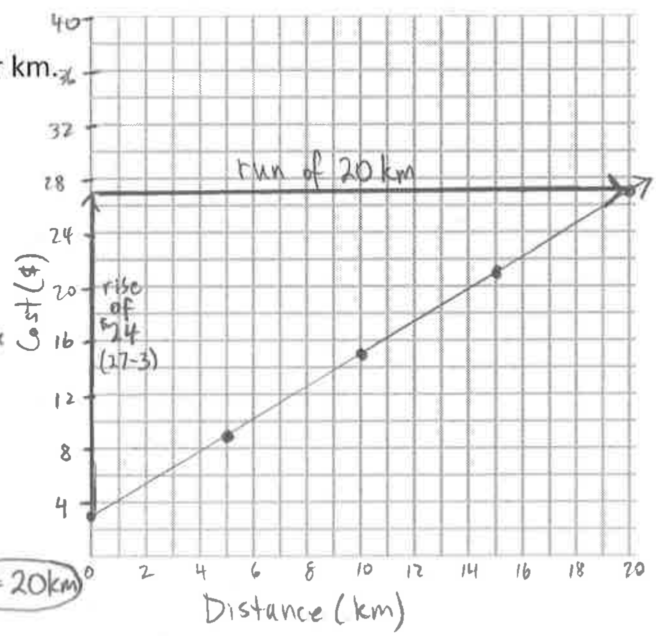
EQUATION: $y = mx + b$ ← y-intercept

cost, slope, distance

$$C = 1.20d + 3$$

a) $C = 1.20(15) + 3$
$= 18 + 3$
$= \$21$

(b) $C = 1.20d + 3$
$27 = 1.20d + 3$
$-3 \quad -3$
$\frac{24}{1.20} = \frac{1.20d}{1.20}$
$d = 20\text{km}$



Example 5: Most cars depreciate as they age. A car costing \$30 000 will have a value of \$2500 at the end of 10 years.

a) Sketch a graph (years on *x-axis*)

b) Find the rate of depreciation using the graph.

c) Write an equation that represents the value, ***V***, at ***t*** years old.

d) Determine the car's value after 4.5 years.

b) $\frac{\text{rise}}{\text{run}} = \frac{-\$27500}{10\text{ yr}} = -\$2750/\text{yr}$

c) $y = mx + b$

thing on y axis, slope, thing on x axis, y-intercept

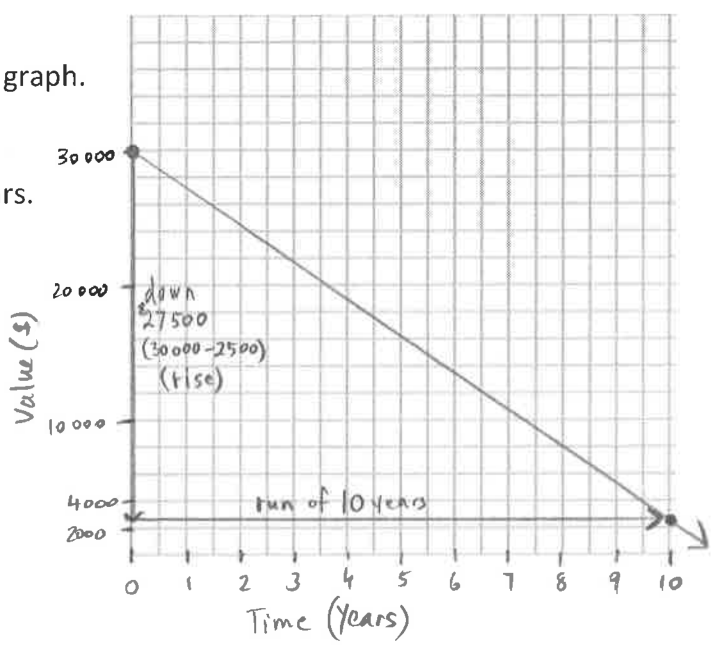
$$V = -2750t + 30000$$

d) $V = -2750(4.5) + 30000$

$V = -12375 + 30000$

$V = \$17625$

(verify on graph)

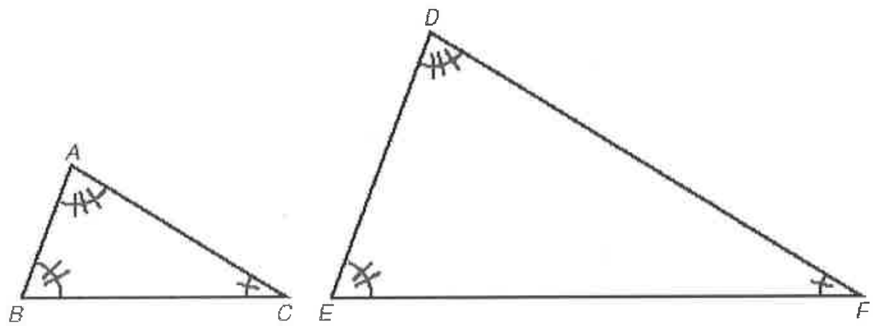


1.3 – Similar Figures

What are '**similar figures**'?

Objects that have the Same shape but different size.

Example 1: Consider the following similar triangles:



What angles 'match up' to each other? What word do we use for this?

$\angle A$ and $\angle D$
$\angle B$ and $\angle E$
$\angle C$ and $\angle F$

We say that $\angle A$ and $\angle D$ are corresponding (same with $\angle B$ and $\angle E$, and also $\angle C$ and $\angle F$)

What do you notice about the corresponding angles?

they are EQUAL (same shape)

What sides are corresponding? Are corresponding sides equal?

AB and DE
AC and DF
BC and EF

NO, corresponding sides are NOT equal (different size)

ratios of corresponding sides are EQUAL!

What is true about the corresponding sides? Let's test the hypothesis:

$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ $\frac{1.9}{3.8} = 0.5$, $\frac{3.4}{6.8} = 0.5$, $\frac{3.6}{7.2} = 0.5$

How do you abbreviate a statement that shows the two triangles as similar?

$\Delta ABC \sim \Delta DEF$

↑ is similar to

What is another term for **similar**? proportional

similar figures stipulations

For two objects to be similar, the following must be true:

1) corresponding angles are EQUAL

2) ratios of corresponding sides are EQUAL

Similar figures in society often have an **original** object or drawing, and then have a **scale diagram** object or drawing.

What is it called if the scale diagram is larger than the original? What are some societal examples of this? billboards, blueprints for computer parts, etc.

enlargement

What is it called if the scale diagram is smaller than the original? What are some societal examples of this? maps, blueprints for buildings, toy models

reduction

Scale Factor

The **scale factor** is the ratio of any two corresponding sides. It is the same for all pairs of similar figures (the sets corresponding sides are in proportion to one another).

$$\text{Scale Factor} = \frac{\text{scale diagram length}}{\text{original length}} = \frac{SC}{OR}$$

A scale factor can be expressed as a FRACTION or a DECIMAL.

If the scale diagram is an **enlargement**, how can you tell by the fraction? Decimal?

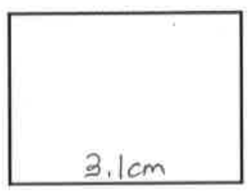
Fraction is improper (numerator larger than denominator) | Decimal is greater than 1

If the scale diagram is a **reduction**, how can you tell by the fraction? Decimal?

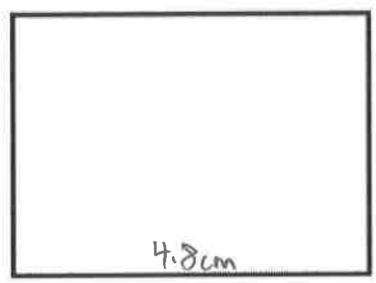
Fraction is proper, decimal is less than 1

Example 2: Find the scale factor for the two similar rectangles.

$\text{Scale Factor} = \frac{SC}{OR} = \frac{4.8\cancel{cm}}{3.1\cancel{cm}} = 1.55$



Original



Scale Diagram

Example 3: Complete the following table:

Original Length	Scale Diag Length	Scale Factor	Enlarge/Reduction
7cm	14cm	$\frac{14}{7} = 2$	enlargement
15m	5m	$\frac{5}{15} = \frac{1}{3}$	reduction
8ft	3.2ft	0.4	reduction
4mm	6mm	1.5	enlargement

$\frac{x}{8} = 0.4 \Rightarrow \frac{\cancel{8}x}{\cancel{8}} = 0.4(8)$

$x = 3.2\text{ft}$

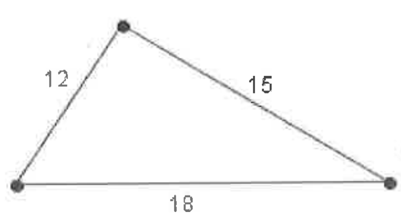
$\frac{6}{y} = 1.5$, $\frac{6}{y} \times \frac{1.5}{1}$

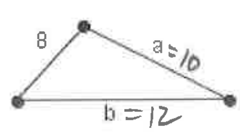
$1.5y = 6(1)$

$\frac{\cancel{1.5}y}{\cancel{1.5}} = \frac{6}{1.5}$

$y = 4$

Example 4: The triangles are similar. Find the unknown sides:





$\frac{12}{8} = \frac{15}{a} = \frac{18}{b}$

$\frac{12}{8} \times \frac{15}{a}$

$12a = 8(15)$

$\frac{\cancel{12}a}{\cancel{12}} = \frac{120}{12}$

$a = 10$

$\frac{12}{8} \times \frac{18}{b}$

$\frac{\cancel{12}b}{\cancel{12}} = \frac{144}{12}$

$b = 12$

Example 5: A scale factor for a chair is $\frac{1}{8}$. If the height of the actual chair is 75cm,
What is the height on the scale drawing?

original

$$\text{Scale Factor} = \frac{SC}{OR}$$

$$\frac{1}{8} = \frac{x}{75}$$

$$\frac{1}{8} \times \frac{x}{75}$$

$$8x = 75$$

$$\frac{\cancel{8}x}{\cancel{8}} = \frac{75}{8}$$

$$x = 9.375\text{cm}$$

Example 6: Find ***x***

$$\frac{YB}{YV} = \frac{YA}{YH}$$

$$\frac{16}{16+22} = \frac{17}{x}$$

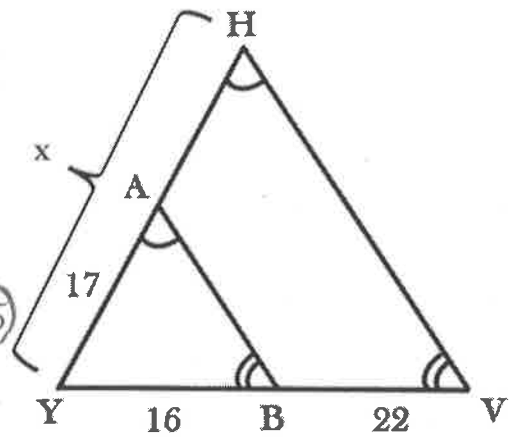
$$\frac{16}{38} \times \frac{17}{x}$$

$$16x = 17(38)$$

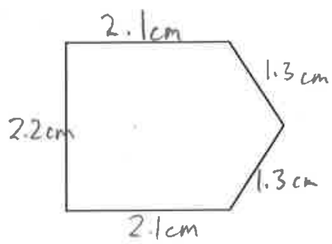
$$16x = 646$$

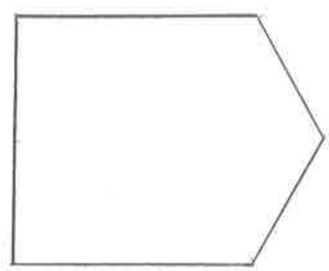
$$\frac{\cancel{16}x}{\cancel{16}} = \frac{646}{16}$$

$$x = 40.375$$



Example 7: Draw a scale diagram with a scale factor of 1.5





$2.1 \times 1.5 = 3.2$cm

$2.2 \times 1.5 = 3.3$ cm

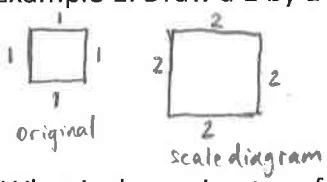
$1.3 \times 1.5 = 2$cm

1.4 – Perimeter, Area, & Surface Area of Similar Figures

Perimeter

Perimeter is the distance around the outside of an object.

Example 1: Draw a 1 by 1 square (the original), and beside it, a 2 by 2 square



What is the scale factor?

$S.F. = \frac{SC}{OR} = \frac{2}{1} = 2$

What is the perimeter of the 1 by 1 square. What is the perimeter of the 2 by 2?

$P_{original} = 1+1+1+1 = 4$

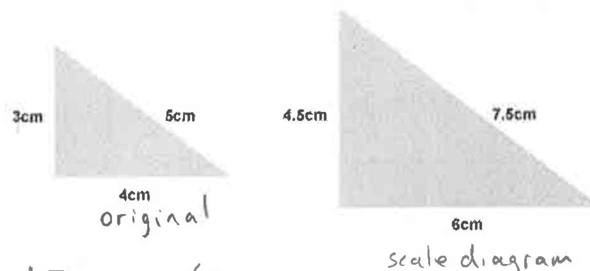
$P_{scaled} = 2+2+2+2 = 8$

What is the ratio of perimeters?

$\frac{8}{4} = \frac{2}{1} = 2$ (same as scale factor!)

Example 2: Find the scale factor if the original is on the left

$S.F. = \frac{SC}{OR} = \frac{4.5}{3} = \frac{1.5}{1} = 1.5$



Now, find the perimeter of each,
& calculate the ratio of perimeters

$P_{original} = 3+4+5 = 12cm$

$P_{scaled} = 4.5+6+7.5 = 18cm$

$\frac{18}{12} = \frac{1.5}{1} = 1.5$ (Same as scale factor!)

Conclusion:

For any set of similar figures, if the scale factor is ***a:b***, then the ratio of perimeters is a : b

Area

Example 3: Find the area of each rectangle in Example 1 and compare the ratio of areas to the scale factor. Then, do the same for Example 2.

Ex 1: $A = lw$

$A_{orig} = (1)(1) = 1$

$A_{scaled} = (2)(2) = 4$

Ratio of Areas $= \frac{4}{1}$ (scale factor is $\frac{2}{1}$)

$S.F. = \frac{2}{1}$

$\left(\frac{2}{1}\right)^2 = \frac{2^2}{1^2}$

$= \frac{4}{1}$ (ratio of areas)

Ex 2: $A = \frac{bh}{2}$

$A_{orig} = \frac{(4)(3)}{2} = 6cm^2$

$A_{scale} = \frac{(6)(4.5)}{2} = 13.5cm^2$

Ratio of Areas $= \frac{13.5}{6} = \frac{2.25}{1}$ (scale factor is $\frac{1.5}{1}$)

$S.F. = \frac{1.5}{1}$

$\left(\frac{1.5}{1}\right)^2 = \frac{1.5^2}{1^2}$

$= \frac{2.25}{1}$ (ratio of areas)

Do the same thing for a 2 by 3 rectangle with a 6 by 9 rectangle:

$A = lw$

$A_{small} = (2)(3) = 6$

$A_{big} = (6)(9) = 54$

$S.F. = \frac{SC}{OR} = \frac{6}{2} = \frac{3}{1} = 3$

ratio of areas $= \frac{54}{6} = \frac{9}{1}$

$\frac{3}{1} \Rightarrow \left(\frac{3}{1}\right)^2 = \frac{3^2}{1^2} = \frac{9}{1}$

Conclusion:

For any set of similar figures, if the scale factor is ***a:b***, then the ratio of areas is $a^2 : b^2$.

Example 4: If the radius of the original circle is 3cm, and the similar circle has a radius of 8cm, find the circumference of each. Does this follow the perimeter rule for similar figures from last page?

$S.F. = \frac{SC}{OR} = \frac{8}{3}$

$C_{orig} = 2\pi r = 2\pi(3) = 6\pi \text{ cm}$

$C_{scale} = 2\pi r = 2\pi(8) = 16\pi \text{ cm}$

ratio of circumference (perim) $= \frac{16\pi}{6\pi} = \frac{8}{3}$

Same as scale factor!

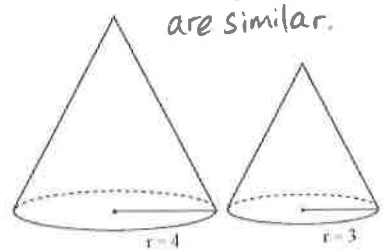
YES, it follows!

The area rule (if scale factor is ***a:b***, area ratio is ***a²:b²***) for similar figures also holds true for ***surface area*** of 3D shapes

Example 5: Find the ratio of the surface area of the larger cone to the surface area of the smaller cone.

$S.F. = \frac{SC}{OR} = \frac{\text{larger}}{\text{smaller}} = \frac{a}{b} = \frac{4}{3}$

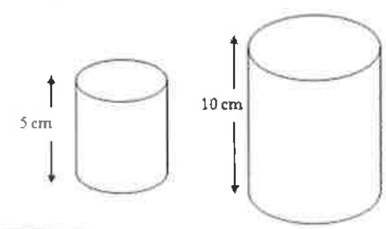
The cones are similar.



ratio of surface areas $= \frac{a^2}{b^2} = \frac{4^2}{3^2} = \frac{16}{9}$ or $16:9$

Example 6: The two cylinders are similar. If the surface area of the largest cylinder is 112π cm², find the surface area of the smaller cylinder.

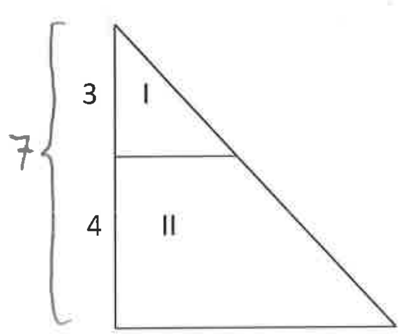
$S.F. = \frac{10}{5} = \frac{2}{1}$

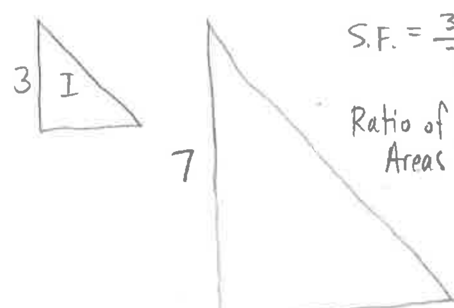


Area ratio $= \frac{2^2}{1^2} = \frac{4}{1}$ or $4:1$

$\frac{4}{1} = \frac{112\pi}{x} \Rightarrow \frac{4x}{4} = \frac{112\pi}{4}$ $\quad x = 28\pi \text{ cm}^2$

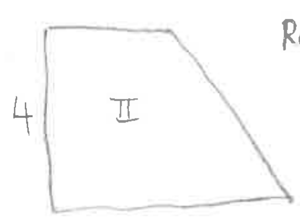
Example 7: Find the ratio of the areas of Figures I and II.





$S.F. = \frac{3}{7}$

Ratio of Areas $= \frac{3^2}{7^2} = \frac{9}{49}$

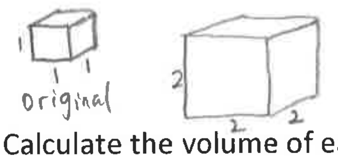


Ratio of Areas $= \frac{3^2}{7^2-3^2} = \frac{9}{49-9} = \frac{9}{40}$

1.5 – Volume of Similar Figures

Example 1: Draw a 1x1x1 cube below (the original). Beside it, draw a 2x2x2 cube. What is the scale factor of the cubes?

$S.F = \frac{SC}{OR} = \frac{2}{1} = 2$



Calculate the volume of each cube. How does the volume ratio compare to the scale factor?

$V = lwh$

$V_{orig} = (1)(1)(1) = 1$

$V_{scale} = (2)(2)(2) = 8$

Volume ratio $= \frac{8}{1}$ $\quad$ $S.F. = \frac{2}{1}$

$\left(\frac{2}{1}\right)^3 = \frac{2^3}{1^3} = \frac{8}{1}$

Example 2: What is the scale factor for a 2x2x2 cube (the original) & a 6x6x6 cube?

$S.F. = \frac{SC}{OR} = \frac{6}{2} = \frac{3}{1} = 3$

Calculate the volume of each cube. How does the volume ratio compare to the scale factor?

$V_{orig} = (2)(2)(2) = 8$

$V_{scale} = (6)(6)(6) = 216$

Volume ratio $= \frac{216}{8} = \frac{27}{1}$ $\quad$ $S.F. = \frac{3}{1}$

$\left(\frac{3}{1}\right)^3 = \frac{3^3}{1^3} = \frac{27}{1}$

Conclusion:

> If the scale factor of two similar solids is ***a:b***, then the ratio of the volumes is $a^3 : b^3$.

Example 3: The ratio of similarity of two cylinders is $\frac{2}{5}$. Find the ratio of their:

a) Perimeters: if S.F is a:b, perim ratio is a:b so $\frac{2}{5}$

b) Areas if S.F. is a:b, area ratio is $a^2 : b^2$, so $\frac{2^2}{5^2} = \frac{4}{25}$

c) Volumes if S.F is a:b, volume ratio is $a^3 : b^3$, so $\frac{2^3}{5^3} = \frac{8}{125}$

Example 4: The surface areas of two spheres are 36 π cm² and 144 π cm². What is the ratio of their volumes?

Ratio of Areas $= \frac{36\pi}{144\pi} = \frac{36 \div 36}{144 \div 36} = \frac{1}{4}$, so $S.F. = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$

so ratio of volumes $= \frac{1^3}{2^3} = \frac{1}{8}$

Example 5: Two similar brass statues are made of Carl Gauss, a great mathematician. The shoe on one statue is 9cm long, and the other is 15cm long. If the volume of the smaller statue is 216cm^3, what is the volume of the larger statue?

$$S.F. = \frac{9^{\div 3}}{15_{\div 3}} = \frac{3}{5}$$

$$\frac{27}{125} = \frac{216}{x}$$

$$x = 1000\text{cm}^3$$

$$\text{Ratio of Volumes} = \frac{3^3}{5^3} = \frac{27}{125}$$

$$27x = (216)(125)$$

$$\frac{27x}{27} = \frac{27000}{27}$$

Example 6: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. What happens to the volume of the cone if:

a) the height is doubled?

$$\text{ratio of volumes} = \frac{\frac{1}{3}\pi r^2(2h)}{\frac{1}{3}\pi r^2 h} = \frac{2\cdot\cancel{\frac{1}{3}\pi r^2 h}}{\cancel{\frac{1}{3}\pi r^2 h}} = \frac{2}{1}$$

Volume will be doubled!

b) the radius is doubled?

$$\text{ratio of volumes} = \frac{\frac{1}{3}\pi(2r)^2 h}{\frac{1}{3}\pi r^2 h} = \frac{\frac{1}{3}\pi 4r^2 h}{\frac{1}{3}\pi r^2 h} = \frac{4\cdot\cancel{\frac{1}{3}\pi r^2 h}}{\cancel{\frac{1}{3}\pi r^2 h}} = \frac{4}{1}$$

Volume is quadrupled!

c) Both height and radius are doubled?

$$\text{ratio of volumes} = \frac{\frac{1}{3}\pi(2r)^2(2h)}{\frac{1}{3}\pi r^2 h} = \frac{\frac{1}{3}\pi 4r^2 2h}{\frac{1}{3}\pi r^2 h}$$

$$= \frac{4\cdot 2\cdot\cancel{\frac{1}{3}\pi r^2 h}}{\cancel{\frac{1}{3}\pi r^2 h}}$$

$$= \frac{4\cdot 2}{1}$$

$$= \frac{8}{1}$$

Volume will be 8 times greater than original