Chapter 6 - Linear Equations

6.1 - Understanding Linear Equations

We continue our investigation of linear equations by working with the fundamentals of algebra. Let's now discuss three mathematics principles before learning how to solve problems by isolating the variable.

The addition and subtraction principle:

For any real number a, b, and c: $a = b$ implies that $a + c = b + c$

And that $a - c = b - c$.

This is to say that if the LS=RS, that manipulating both sides by the same number and action, will keep the equation statement true.

Example 1) $4 + 3 = 7$

Addition: $(4+3)+10=(7)+10 \Rightarrow 17 = 17$

Subtraction: $(4+3)-5=(7)-5 \Rightarrow 2 = 2$

Example 2) $x - 5 = 4$

$+5 \quad +5$

$\boxed{x=9}$

Example 3) $3x - 4 = 2x + 1$

$+4 \quad +4$

$3x = 2x + 5$

$-2x \quad -2x$

$1x = 5 \Rightarrow \boxed{x=5}$

The multiplication and division principle

For any real number a, b, and c: $a = b$ implies that $a \times c = b \times c$

And that $a \div c = b \div c$.

This is to say that if the LS=RS, that manipulating both sides by the same number and action, will keep the equation statement true.

Example 1) $5 + 3 = 8$

Multiplication: $2 \times (5 + 3) = 8 \times 2 \Rightarrow 10 + 6 = 16 \Rightarrow 16 = 16$

Division: $(5 + 3) \div 2 = 8 \div 2 \Rightarrow 4 = 4$

Example 2) $4x = 24$

$\frac{4x}{4} = \frac{24}{4}$

$\boxed{x = 6}$

Example 3) $6x + 3 = 2x - 9$

$-2x \quad -2x$

$4x + 3 = -9$

$-3 \quad -3$

$\frac{4x}{4} = \frac{-12}{4}$

$\boxed{x = -3}$

Reciprocals

Two numbers are reciprocals of one another when their product equals one. The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ because $\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1$

Example: State the reciprocal of the following numbers:

a) $\frac{3}{4} \Rightarrow \frac{4}{3}$

b) $\frac{1}{4} \Rightarrow \frac{4}{1} = 4$

c) $-4 \Rightarrow \frac{-4}{1}$

$\Rightarrow \frac{-1}{4}$

d) $\frac{-2}{x} \Rightarrow -\frac{x}{2}$

Example 1) $\frac{4}{5}x = 12$

$5 \times \frac{4}{5}x = 12 \times 5$

$\frac{4x}{4} = \frac{60}{4}$

$\boxed{x = 15}$

Example 2) $\frac{-3}{4}y = -6$

$4 \times \frac{-3}{4}y = -6 \times 4$

$\frac{-3y}{-3} = \frac{-24}{-3}$

$\boxed{y = 8}$

Example 3) $\frac{3y}{2} - 1 = 8$

$+1 \quad +1$

$2 \times \frac{3y}{2} = 9 \times 2$

$\frac{3y}{3} = \frac{18}{3}$

$\boxed{y = 6}$

Golden Rule:
To isolate a variable, you must do the EQUAL and OPPOSITE action to what is currently locking the variable up. That equal-and-opposite principle applies not only to each individual step, but also to the overall order of operations as well.

When you don't have any variables, use (past):	When you have a variable you are trying to solve, use (present):
B **E** **D** **M** **A** **S**	**S** **A** **M** **D** **E** **B**

Math and English Translations

A fundamental skill for applying math to the real world is the ability to translate your ideas (from your primary language) into a mathematical (algebraic) language. Let's observe some key terms and phrases that are the building blocks in our mathematics lexicon.

Addition (sum, plus, increase, total):
The sum of a number and five $x+5$
Six plus a number $6+y$
A number is increased by eight $z+8$
The total of two numbers is negative twelve $x+y=-12$

Subtraction (difference, minus, less, deduct, decrease) :
The difference between a number and five $x-5$
The difference between five and a number is seven $5-y=7$
A number is decreased by one $n-1$

5

Multiplication (product, times, of, by):

Fifteen percent of a number	$.15n$
How many squares are on the face of a three by three Rubic's cube?	$3 \times 3 = r$
Twice a number	$2n$
The product of a number and seven	$7y$
Half a number is three	$\frac{1}{2}x = 3$

Division (quotient, goes into, how many times):

The quotient of a number and four is five	$\frac{x}{4} = 5$
The quotient of four and a number is two	$\frac{4}{y} = 2$
How many times does four go into ten	$\frac{10}{4} = w$
Half a number is three	$\frac{x}{2} = 3$

Examples:

1) The sum 5 and three consecutive odd integers is 20, what are the integers?

1 2 3 4 5 6 . . .

x $\quad x+2$ $\quad x+4$

$5+(x)+(x+2)+(x+4) = 20$

$3x + 11 = 20$

$3x = 9$

$\boxed{x = 3}$

$x+2 = \boxed{5}$

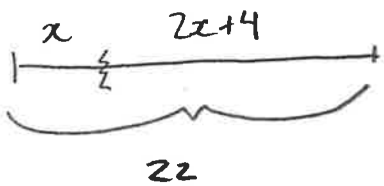
$x+4 = \boxed{7}$

Check:

$5+3+5+7 = 20$

$20 = 20$ ✓

2) A 22cm string is cut into two pieces. One piece is 4cm longer than twice the length of the other piece. Find the lengths of the two pieces.



$$(x)+(2x+4)=22$$

$$3x+4=22$$
$$-4 \qquad -4$$
$$3x=18$$
$$\boxed{x=6}$$

$$2(6)+4$$
$$12+4=\boxed{16}$$

$$6+16=\boxed{22}$$ ✓

3) Extension problem: The sum of two numbers is 5; their difference is 1. What are the two numbers?

$$x+y=5$$
$$x-y=1 \Rightarrow x=y+1$$

$$(y+1)+y=5$$
$$2y+1=5$$
$$2y=4$$
$$\boxed{y=2}$$

$$x=2+1$$
$$\boxed{x=3}$$

Check: $2+3=5$

$3-2=1$ ✓

6.2 - Removing Fractions, Decimals, and Parentheses

There are a few different methods for how you can approach solving algebraic equations. But the golden rule (outlined in 6.1) can strategize your decisions.

Here are a few other tips to help guide you:

i) The end goal is to have a number on one side of the equal sign, solving for the variable on the other.

ii) Whole numbers are easier to solve than fractions, but fractions are easier to solve than decimals.

iii) To convert decimals to fractions, write the decimal over 1; then multiply numerator and denominator by a factor of ten depending on the place value of the last digit of the original decimal.

iv) To convert between fractions and whole numbers, multiply the entire equation by a common denominator for all of the fractions in the equation.

v) Remember the golden rule for algebra: To solve (reorganize) the equation, follow SAMDEB and do the equal and opposite actions along the way to isolate your wanted variable.

Examples:

1) $\frac{1}{3}x - \frac{5}{6} = 4 + \frac{1}{2}x \quad \times 6$

$$6\left(\frac{1}{3}x - \frac{5}{6}\right) = \left(4 + \frac{1}{2}x\right)6$$

$$2x - 5 = 24 + 3x$$
$$-2x \qquad\qquad -2x$$

$$-5 = 24 + x$$
$$-24 \quad -24$$

$$\boxed{x = -29}$$

2) $0.001y + .02 = y - .003$

$$1000 \times \frac{1}{1000}y + \frac{2}{100} = y - \frac{3}{1000} \quad \times 1000$$

$$y + 20 = 1000y - 3$$

$$\frac{23}{999} = \frac{999y}{999}$$

$$\boxed{\frac{23}{999} = y}$$

3) $.04x + .5(x - 2) = \frac{1}{2}(x + 5)$

$$100 \times \frac{4}{100}x + \frac{1}{2}(x-2) = \frac{1}{2}(x+5) \quad \times 100$$

$$4x + 50(x-2) = 50(x+5)$$

$$4x + 50x - 100 = 50x + 250$$
$$-50x \qquad\qquad -50x$$

$$4x - 100 = 250$$
$$+100 \quad +100$$

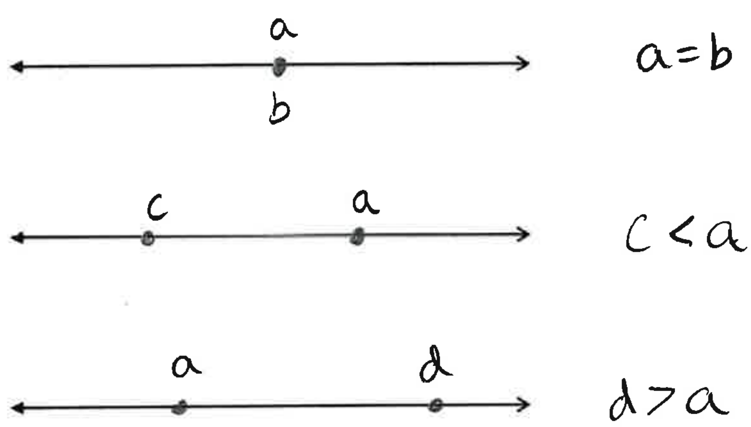
$$\frac{4x}{4} = \frac{350}{4}$$

$$x = \frac{175}{2} = 87.5$$

$$\boxed{x = 87.5}$$

6.3 - Linear Inequalities

Up until this point, we have almost exclusively discussed when a variable is equal to a value. Comparing two values isn't just important when two values are equal, but also when they are unequal. Using a number line, and starting from point *a*, a new value is either equal to *a* (point *b*), less than *a* (point *c*), or greater than a (point *d*).



<u>Inequality Symbols</u>

Symbol	Definition	Example
$=$	equal to	$3 = 3$
$\neq$	not equal to	$2 \neq 4$
$<$	less than	$6 < 8$
$\leq$	less than or equal to	$-6 \leq 0$ $5 \leq 5$
$>$	greater than	$7>3$
$\geq$	Greater than or equal to	$-2 \geq -4$ $5 \geq 5$

Note: The "or" aspect is known as an "inclusive or." The statement is true if either the inequality or the equality is true.

Graphing inequalities

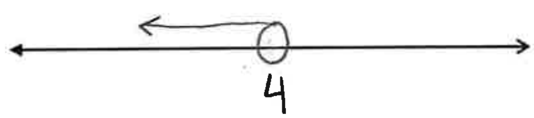
When we are solving algebraic inequalities, we will often get a solution that will take the form of $x < 4$ (x is less than 4). What are some possible values of x that make this true? 0, -1100, 1.5, 3.999999 for example, are all values that make the statement true. Thus, as we can see, there are an infinite number of values that are less than 4; thus, graphing our solutions can represent this well.

Note: There are two types of dots that we will use to represent the value of a number.

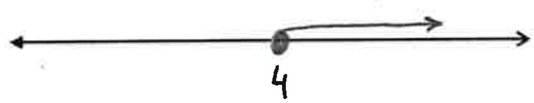
Open dot: ○ Used with less (<) than and greater (>) than inequalities
Closed dot: ● Used with less than or equal to ($\leq$) and greater than or equal to ($\geq$)

Examples:

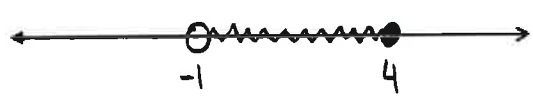
a) $x < 4$



b) $x \geq 4$



c) $4 \geq y > -1$



d) $3 < y \leq -4$

Translating to Inequalities

Words	**Example**	**Mathematical Statement**
At most	Classes will have at most thirty students	$c \leq 30$
At least	You must get at least 50% to pass our course	$g \geq 50$
Must not exceed	In a school zone, you must not exceed 30km/h.	$s \leq 30$
Must Exceed	To ride on the roller coaster your height must exceed four feet.	$h > 4$
Less than	A cat's life's expectancy is less than a dog's	$c < g$
More than	If you work more than 40 hours a week you are entitled to overtime pay.	$h > 40$

Note: It is the convention to write the variable on the left side of the inequality. When switching the direction, make sure to have the inequality open to the same value.

For instance, $-4 > x$ becomes $x < -4$

$5 < y$ becomes $y > 5$

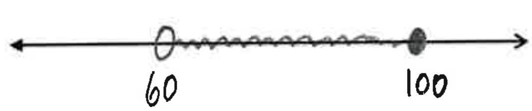
Examples: Translate the statement, then graph the inequality

a) In BC, your working wage must be at least \$13.85/hr.

$w \geq 13.85$

b) Unless otherwise posted, your speed on highways must exceed 60km/h and can be no greater than 100km/h.

$60 < s \leq 100$



6.3 - Adding and Subtracting Linear Inequalities

Silver Rule or algebra: What you do to one side, you have to do to the other.

The Addition Principle of Inequalities

For any real numbers *a*,*b*, *c*:

$a < b = a + c < b + c$ & $a < b = a - c < b - c$ also true for $\leq$

$a > b = a + c > b + c$ & $a > b = a - c > b - c$ also true for $\geq$

Note: Solve inequalities (=), exactly as you would regular equalities

For instance: $5 > 1$
$= 5 + 2 > 1 + 2$
$= 7 > 3$

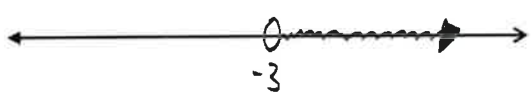
$-6 < 4$
$= -6 - 3 < 4 - 3$
$= -9 < 1$

Examples: Solve and graph

a) $x + 5 > 2$

$-5 \quad -5$

$x > -3$



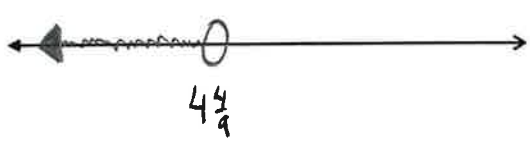
b) $5x - 1 < .5x + 19$

$+1 \quad\quad +1$

$5x < \frac{1}{2}x + 20$

$-\frac{1}{2}x \quad -\frac{1}{2}x$

$4\frac{1}{2}x < 20$



$\frac{9}{2}x < 20$

$9x < 40$

$x < \frac{40}{9} = 4\frac{4}{9}$

Special Cases

Periodically we encounter problems where the solution has no variables. These important expressions appear often throughout mathematics, regardless of the complexity of the problem.

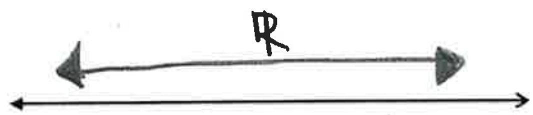
Tautology: $5x + 5 - 2x < 3x + 8$

$3x + 5 < 3x + 8$

$-3x \qquad -3x$

$5 < 8$ ✓ Always true!

$x \in \mathbb{R}$



Check: $x = 1$

$5(1) + 5 - 2(1) < 3(1) + 8$

$5 + 5 - 2 < 3 + 8$

$8 < 11$ ✓

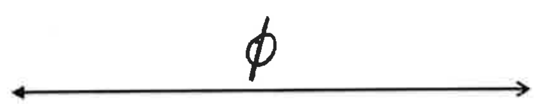
Empty Set: $4x - 1x - 6 > 3x - 2$

$3x - 6 > 3x - 2$

$-3x \qquad -3x$

$-6 > -2$ ✗ ϕ

$x \in \phi$



Check: $x = 1$

$4(1) - 1(1) - 6 > 3(1) - 2$

$4 - 1 - 6 > 3 - 2$

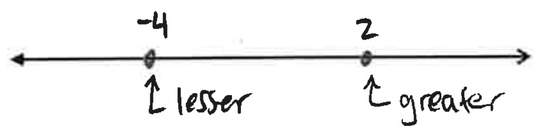
$3 - 6 > 3 - 2$

$-3 > 1$ ✗

Note 1: Be careful around negative solutions:

i) $-3 > -11$

ii) $-4 < 2$



Note 2: When you multiply or divide the statement by a negative (-1 in most cases) the inequality sign flips $x < 4 = -x > -4$

For example

flipped

$-5 < 3 \quad \rightarrow \quad 5 > -3$

X-1 X-1

flipped

$\frac{-x}{-1} > \frac{7}{-1} \quad \rightarrow \quad x < 7$

6.5 - Multiplying Linear Inequalities

The Multiplication Principle of Inequalities

For any positive real numbers *a,b, c*:

$a < b = ac < bc$ & $a < b = a(-c) > b(-c)$ also true for $\leq$

$a > b = ac > bc$ & $a > b = a(-c) < b(-c)$ also true for $\geq$

For instance: $5 > 1$

$= 5 \times 2 > 1 \times 2$

$= 10 > 2$

$-6 < 4$

$= -6(-3) < 4(-3)$

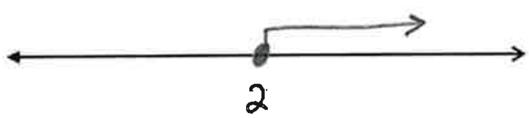
$\neq +18 <- 12$ ***false***

$= 18 >- 12$ ***true***

Examples: Solve and graph

a) $\frac{6x}{6} \geq \frac{12}{6}$

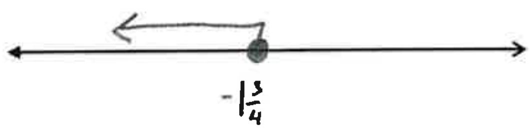
$x \geq 2$



b) $\frac{-4x}{-4} \geq \frac{7}{-4}$

$x \leq \frac{-7}{4}$

$x \leq -1\frac{3}{4}$



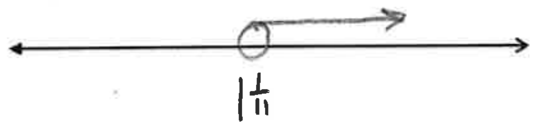
12x

c) $-\frac{2}{3}x < \frac{1}{4}(x-4)$ $\times 12$

$-8x < 3(x-4)$

$-8x < 3x - 12$

$\frac{-11x}{-11} < \frac{-12}{-11}$



$x > \frac{12}{11}$

$x > 1\frac{1}{11}$

d) $-3(x-2)+2x < -4x(2-1)$

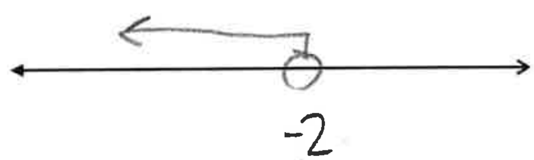
$-3x+6+2x < -4x(1)$

$-x+6 < -4x$

$+4x \quad\quad +4x$

$3x+6 < 0$

$-6 \quad -6$



$\frac{3x}{3} < \frac{-6}{3}$

$x < -2$

6.6 - Applications of Linear Equations

In this section, we will take the fundamental skills we learned throughout the unit and apply them to more real-world examples.

Charts and graphs enable us to make findings without having to know every single value. Take the following scenario:

When Rachel was 10 years old, her net worth was \$50. When she turned 15, her net worth was \$550. Assuming linear growth,

a) What was her net worth when she was 13?

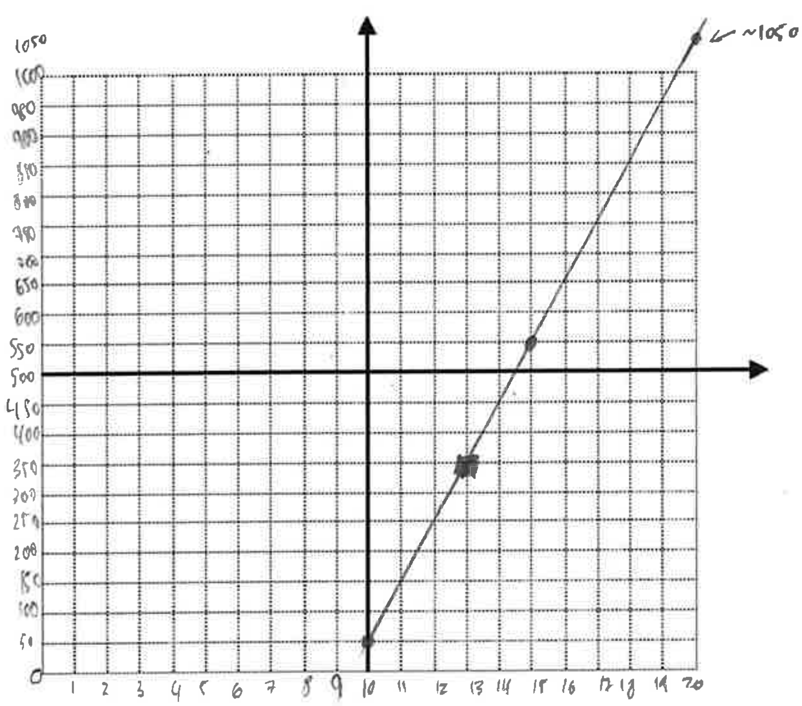
b)How much will her net worth be at 20?

For part a) because the year we are looking for is between two points that we already know, the solution is called *interpolation.*

For part b) because we are looking for a year that is beyond any one of our known points, the solution is called *extrapolation.*

a) interpolate: at 13, \$=350

b) extrapolate: at 20, \$ = 1050



Example: Paul was a foot and a half tall when he was born and is five and a half feet tall when he turned 16; assuming linear growth, how tall was he when he turned 8 and how tall will he be when he turns 50?

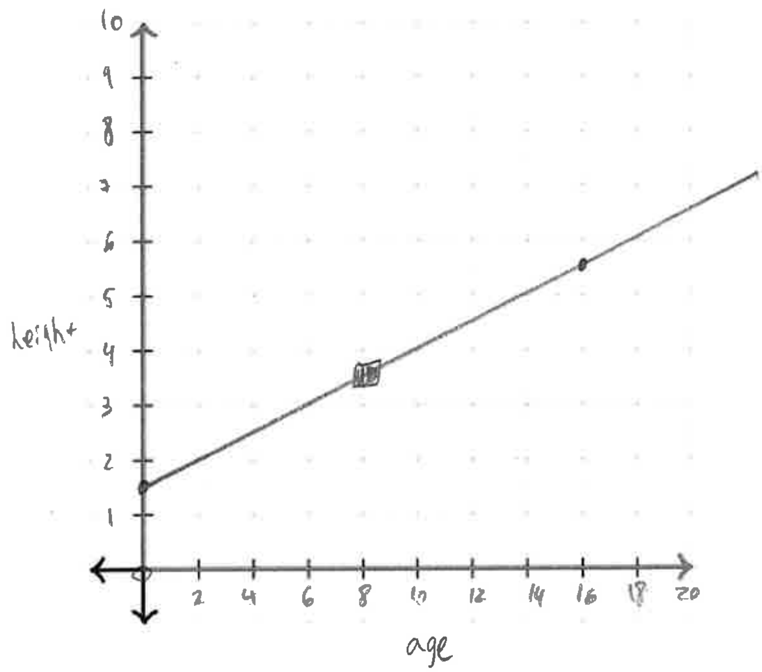
At 8, height = 3.5 ft.

At 50, off the chart!

$$\frac{y_2-y_1}{x_2-x_1} = \frac{5.5-1.5}{16-0}$$

$$= \frac{4}{16} = \boxed{\frac{1}{4}}\frac{\text{ft}}{\text{year}}$$

↑
m



$b = 1.5$

$y = mx + b$

$y = \frac{1}{4}x + 1.5$

$y = \frac{1}{4}(50) + 1.5$

$y = 12.5 + 1.5$

$\boxed{y = 14 \text{ ft!}}$

Not all scenarios can be demonstrated linearly!

Notes: