Chapter 5 - Polynomials

5.1 - Defining Polynomials

Variable

A variable is a letter whose value is an unknown real number.

Examples: $a, b, c, \ldots x, y, z, \alpha, \beta, \theta$

Term

A term is a number, or product of a number and variable(s) raised to a power.

Examples: $3, \ -2x, \ -5x^2, \ 4x^2y^4z$

Coefficient

The coefficient of a term is the numerical factor.

Examples:

Term	Coefficient
3	3
$-2x$	-2
$\frac{x^4}{5} = \frac{1}{5}x^4$	$\frac{1}{5}$
$-2.4ab$	-2.4
$x = 1x$	1
$-y = -1 \times y = -1y$	-1

Note: If a term is a number only, it is called a *constant*.

Monomial

A monomial is a single termed expression of the type cx^n, where c is a real number coefficient, x is a variable, and n is a whole number exponent.

Monomial examples (whole number exponents)	Non-monomial examples (non-whole number exponents)
$3xy^2$	$\frac{3}{xy^2}$
$\frac{1}{2} = \frac{1}{2}x^0$	$a^{1/2}$
$-wz$	wz^{-1}
$\sqrt{2}x$	$\sqrt{2x}$

Polynomial

A polynomial is a monomial or a combination of sums and/or differences of monomials.

Type	Example
Monomial (one term)	$2,\ 3x,\ -4x^3,\ -6xy,\ 0,\ 4^2b^2$
Binomial (two terms)	$2+3x,\ -4x^3-6xy$
Trinomial (three terms)	$-4x^3-6xy+2$
Polynomials (multiple terms)	$-4x^3-6xy+3x+2$

Degree of a Polynomial

The degree of a term in a polynomial is the sum of the exponents for the variables in that term ($5xy^3$ => has a degree of 4). Each term has a degree, but the biggest one is the most important. The term that has the biggest, most important degree, also stands for and is known as the "degree of the polynomial" itself.

Polynomial	Degrees of terms	Degree of Polynomial
$4x^2 + 3x - 1$	$4x^2 = 2$ $3x^1 = 1$ $-1x^0 = 0$	As $4x^2$, which has degree 2, is the largest of the polynomial, the polynomial itself is said to have a degree of 2.
$5x^2y + 6x^3yz$	$5x^2y^1 = 3$ $6x^3y^1z^1 = 5$	The term with the largest polynomial is $6x^3yz = 5$, so the degree of the entire polynomial is 5.

Leading Term

It may seem a bit redundant, but there is one last classification of polynomials we must discuss, the "leading term." The Leading Term is the term with the highest degree. For the examples in the box above, the polynomial $4x^2 + 3x - 1$ has a leading term of $4x^2$ and the polynomial $5x^2y + 6x^3yz$ has a leading term of $6x^3yz$.

So to recap: Polynomials are made up of terms. Each term has its own degree. The term that contains the highest degree is known as the leading term. The entire polynomial has a degree, which is equal to the degree of the leading term.

Putting it all together:

Polynomial	$-3x^2y-5xyz^4+y-\frac{1}{4}$
Terms	$-3x^2y,\ -5xyz^4,\ y,\ -\frac{1}{4}$
Coefficients	-3, -5, 1, $-\frac{1}{4}$
Degree of each term	3, 6, 1, 0
Leading Term	$-5xyz^4$
Degree of polynomial	6
ReWritten in Descending Order	$-5xyz^4-3x^2y+y-\frac{1}{4}$

Note: A polynomial is usually written in descending order of powers. For example, the polynomial $3a+5-2a^2$ is rewritten as $-2a^2+3a+5$. A polynomial written with more than one variable should also be written in alphabetical order. For example, the polynomial $xy+x^2-y^2$ is rewritten as x^2+xy-y^2 since the degree of each term are the same, we break the tie alphabetically.

Like Terms

"Like terms" are terms that have the SAME VARIABLES raised to the SAME POWER (keep in mind that the order of the variables does not matter). If two or more terms are "like" one another, they can be "combined" or "simplified" via addition or subtraction. When the terms are combined, only the coefficients will add or subtract, the variables, and powers will stay the same!

Term	Like Terms? (Y/N)	Simplified
$4x^2 - 3x$	NO (different exponents)	$4x^2 - 3x$
$5y^3 + 4y^3$	YES (same V, same E)	$9y^3$
$3x^2y + 3xy^2$	NO (different E)	$3x^2y + 3xy^2$
$\frac{1}{4}zx + \frac{3}{4}xz$	YES (same V, same E)	$1xz = xz$
$3zx^2 - 2yx^2$	NO (different V)	$3zx^2 - 2yx^2$

Simplify the expression: $10x^2y - 5x^2 + 3x^2y + 7z^8 + 3x^2$

1) Rewrite the question: $10x^2y - 5x^2 + 3x^2y + 7z^8 + 3x^2$
2) Collect the like terms together: $10x^2y + 3x^2y - 5x^2 + 3x^2 + 7z^8$
3) Combine like terms: $13x^2y - 2x^2 + 7z^8$

Simplify: $8 + 5t - 7ty - 4 - 6t + y + 3y + 1t - 2 + 10y$

1) $8 + 5t - 7ty - 4 - 6t + y + 3y + 1t - 2 + 10y$
2) $-7ty + 5t - 6t + 1t + y + 3y + 10y + 8 - 4 - 2$
3) $-7ty + 14y + 2$

5.2 - Adding and Subtracting Polynomials

To add polynomials, we can use two approaches, solving horizontally or solving vertically. In both cases, arranging like terms together will help keep your work organized and save potential computational errors.

Note: You will not be required to learn Algebra Tiles in our course. You will not be tested on them during assessments.

Adding Polynomials Vertically (less common approach)
To add polynomials vertically, place the polynomials with like terms in the same columns. If one polynomial lacks a term that is possessed by the other, a blank space is left at that location.

Example: $(-3x^2 - 2) + (4x^2 + 5x - 1)$

$$\begin{array}{r} -3x^2 + 0x - 2 \\ +\quad 4x^2 + 5x - 1 \\ \hline x^2 + 5x - 3 \end{array}$$

Adding Polynomials Horizontally (more common approach)
To add polynomials horizontally collect like terms, and simplify.

Example: $(2x^2 - 5) + (-3x^2 + 2x + 2)$

$-3x^2 + 2x^2 + 2x - 5 + 2$

$-x^2 + 2x - 3$

Subtracting Polynomials Horizontally (more common approach)

To subtract an entire polynomial (one that is in brackets) we must apply the negative to each and every term in that polynomial. To do this, we will multiply each term by -1, which changes the sign of each individual term.

Consider: $-(3x^2 - 4x + 9) \Rightarrow -3x^2 + 4x - 9$

Example: $(2x^3 + 3x^2 - x + 6) - (-2x^3 + 3x^2 + 4x - 11)$

$2x^3 + 3x^2 - x + 6 + 2x^3 - 3x^2 - 4x + 11$

$2x^3 + 2x^3 + 3x^2 - 3x^2 - x - 4x + 6 + 11$

$4x^3 - 5x + 17$

Subtracting Polynomials Vertically (less common approach)

To subtract, a polynomial vertically, arrange the polynomials in descending order of powers; multiply the subtraction (a negative sign) throughout the entirety of the appropriate polynomial; then ADD the polynomials together (essentially we have used up the negative and now combine our two polynomials via addition.

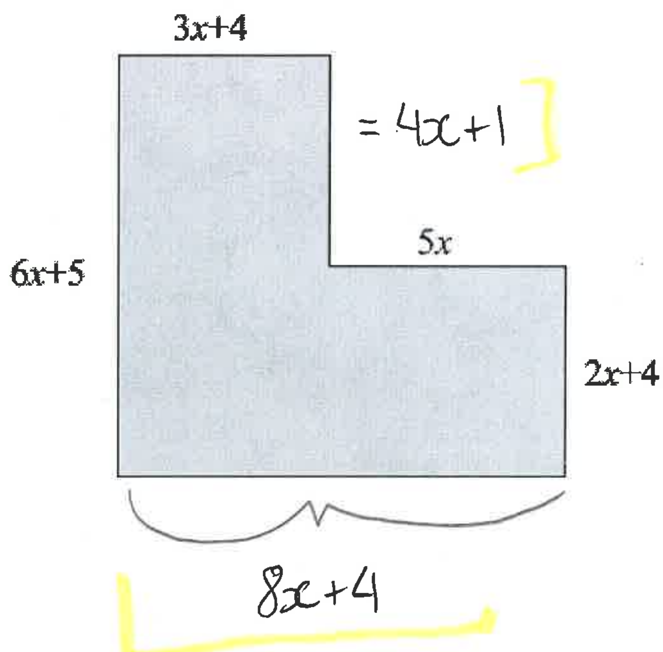
Example: $(3x^3 - 4x^2 + 2x + 5) - (2x^3 - 4x^2 + 8x - 1)$

$$\begin{array}{r} 3x^3 - 4x^2 + 2x + 5 \\ + \; -2x^3 + 4x^2 - 8x + 1 \\ \hline x^3 - 6x + 6 \end{array}$$

Perimeter Problems

We can now solve geometric problems centred around finding the perimeter of an object. Recall that the perimeter measures the path that surrounds a two-dimensional object -- the total distance of "walking around the outside of the shape"

Example: Determine the perimeter of the figure (lengths are in cm).



$6x+5+3x+4+4x+1+5x+2x+4+8x+4$

$6x+3x+4x+5x+2x+8x = 28x$

$+$

$5+4+1+0+4+4 = 18$

$28x+18$

5.3 - Multiplying Polynomials

Multiplying Monomials

To multiply two monomials together, remember that "like gets with like." Multiply the numerical factors together, then multiply the variable factors.

a) $(3y^4) \times (5y^2)$
$= (3 \times 5)(y^4 \times y^2) = (15)(y^{4+2)} = (15)(y^6) = 15y^6$

b) $(-4xyz^2)(-3x^2yw)$
$= (-4)(-3)(x)(x^2)(y)(y)(z^2)(w) = 12(x^{1+2})(y^{1+1})(z^2)(w^1)$
$= (12)(x^3)(y^2)(z^2)(w) = (12wx^3y^2z^2)$

c) $(2x)(2y)(2xy)(2xyz)$
$= (2)(2)(2)(2)(x)(x)(x)(y)(y)(y)(z) = 16x^{1+1+1}y^{1+1+1}z^1 = 16x^3y^3z^1$

Multiplying Polynomials by Monomials

To multiply a polynomial by a monomial, we must utilize the distributive property to remove the parentheses, then simplify. The essence of the distributive property is to make sure that each term in one set of brackets is worked upon by each term in the other set of brackets.

The Distributive Property

$$a(b+c) = (a \times b) + (a \times c) = ab + ac$$

Eg. Multiply the polynomials together.

a) $4(2y + 3)$

$= (4 \times 2y) + (4 \times 3) = 8y + 12$

b) $6x^2(3x - 4y^2)$

$= (6x^2 \times 3x) - (6x^2 \times 4y^2) = 18x^3 - 24x^2y^2$

c) $5(2 + 3)$

$= (5)(2) + (5)(3) = 10 + 15 = 25$ OR $5(2+3) = 5\ (5) = 25$

Multiplying Polynomials by Polynomials*

To multiply multi-termed polynomials together, use the distributive property. The acronym FOIL (First, Outside, Inside, Last) can help you make sure that all your terms are accounted for.

a) $(2x + 3)(6x - 4)$

First *Outside* *Inside* *Last*

$= (2x)(6x) + (2x)(-4) + (3)(6x) + (3)(-4)$

$= 12x^2 - 8x + 18x - 12$

$= 12x^2 + 10x - 12$

b) $(4x - 3)(2x + 1)$

First *Outside* *Inside* *Last*

$= (4x)(2x) + (4x)(1) + (-3)(2x) + (-3)(1)$

$= 8x^2 + 4x - 6x - 3$

$= 8x^2 - 2x - 3$

5.4 - Dividing Polynomials

Dividing polynomials follows the same principle that "like divides like." Numbers will only simplify other numbers, and same-letter variables will simplify same-letter variables.

Dividing a Polynomial by a Constant

To divide a polynomial by a constant, divide each term of the polynomial by the constant.

Theory: $\frac{a+b+c}{d} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d}, d \neq 0$

Example:

a) $\frac{10x^2+5x-20}{5} = \frac{10x^2}{5} + \frac{5x}{5} - \frac{20}{5} = 2x^2+x-4$

b) $\frac{-8y^3-24y^2+2}{-4} = \frac{-8y^3}{-4} + \frac{-24y^2}{-4} + \frac{2}{-4}$

$= 2y^3+6y^2-\frac{1}{2}$

Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial (remember, like divides like).

c) $\frac{7x^3-x^2+x}{x} = \frac{7x^3}{x} - \frac{x^2}{x} + \frac{x}{x}$

$= 7x^2-x+1$

d) $\frac{10x^2y-20xy}{-5x}$ $= \frac{10x^2y}{-5x} + \frac{-20xy}{-5x} = -2xy + 4y$

e) $\frac{-9x^2y+3x^5y^2}{-3x^2y}$ $\frac{-9x^2y}{-3x^2y} + \frac{3x^5y^2}{-3x^2y} = 3 + (-1x^3y)$

$= -x^3y + 3$

Putting it all together

f) $-\frac{(4x^2-3x+6)}{2} - 2x(-x+1)$ $= -\frac{4x^2}{2} + \frac{3x}{2} + \frac{6}{2} + 2x^2 - 2x$

$= -2x^2 + \frac{3x}{2} + 3 + 2x^2 - 2x$

$= \frac{3x}{2} - 2x + 3 = \frac{3x}{2} - \frac{4x}{2} + 3$

$= \boxed{\frac{-x}{2} + 3}$

g) $\frac{16x^2-12x}{4x} + \frac{8x(1-x)}{2x}$

$\frac{4x(4x-3)}{4x} + \frac{8x(1-x)}{2x}$

$4x-3+4(1-x) = 4x-3+4-4x$

$= 4x-4x-3+4 = \boxed{1}$

Notes: