Chapter 3 - Rational Numbers

3.1 - Defining Rational Numbers

As discussed in the previous unit, Rational Numbers are numbers that can be written as a fraction (where the denominator does not equal 0). Furthermore, a rational number is one that can be expressed as a terminating or repeating decimal. Thus, fractions and decimals can hold the same value - they may just look different.

For example:

$\frac{1}{4} = .25$] $\frac{3}{8} = .375$]

$\frac{2}{9} = .\overline{2}$] $\frac{7}{11} = .\overline{63}$]

$\pi = 3.14159...$] $e = 2.71828...$]

More formally, a real number x is rational if $x = \frac{a}{b}$, with a and b both being integers, and $b \neq 0$.

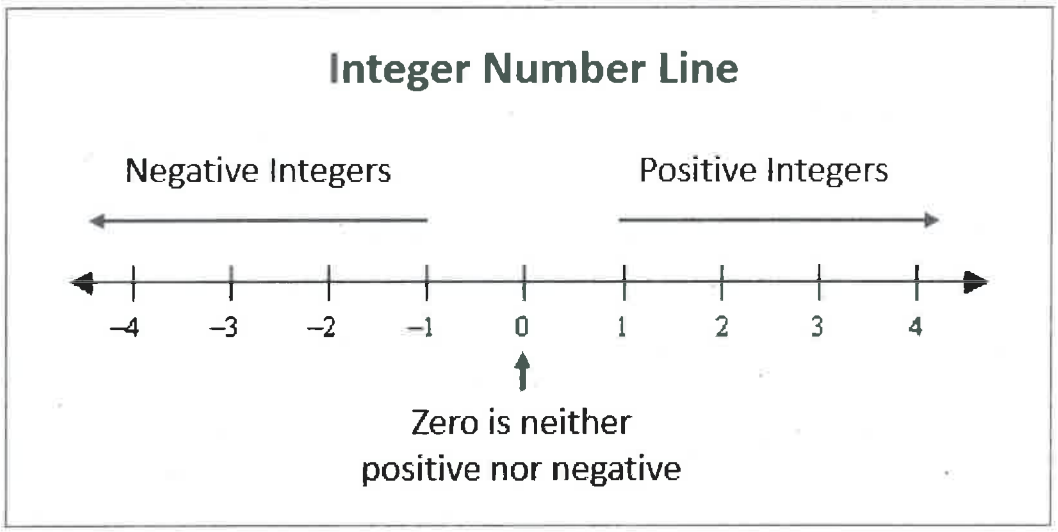
Keep in mind that all integers can be written as fractions because they all can be written with a denominator of 1. In fact, decimals are just numbers that have a denominator with some power of 10.
For example:

$5 = \frac{5}{1}$] $-2 = \frac{-2}{1}$]

$.4 = \frac{.4}{1} = \frac{4}{10}$] $1.635 = \frac{1.635}{1} = \frac{1635}{1000}$]

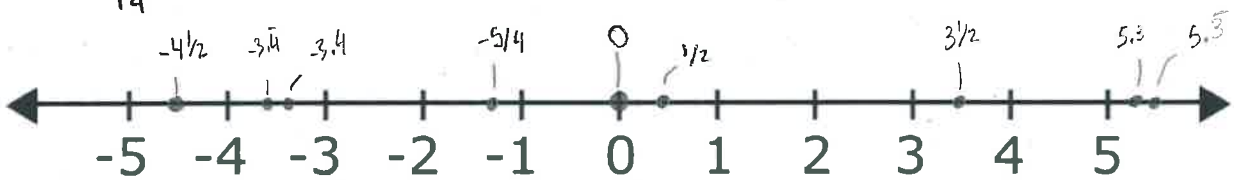
Remember, when trying to convert fractions to decimals, ask yourself "how many times does the denominator fit inside the numerator." For example, $\frac{6}{3} = 2$ because 3 fits inside of 6 exactly 2 times. $\frac{3}{6} = \frac{1}{2} = .5$ because only half of 6 can fit inside of 3 (and half of 2 fits inside of 1).

As we delve into more complicated number systems, determining which numbers are greater than one another becomes less obvious. A number line can help with this:



Example: On the number line below, locate the following numbers:
0, ½, -5/4, 5.3, $5.\overline{3}$, 3½, $-4\frac{1}{2}$, -3.4, $-3.\overline{4}$

$= -1\frac{1}{4}$



Larger numbers exist to the RIGHT of smaller numbers on the number line. Numbers that are more negative are considered "smaller" or "lesser" and exist to the LEFT. When in doubt, it can be very useful to think about money and debt: which value would you rather have to represent your bank account -- that number will lie to the right.

Inequalities

When comparing numbers to see which is greater or lesser than one another we use symbols of inequality.

Less than:

$<$, when read from left to right means less than.

Eg. $3 < 5$ or $-5 < -3$

Described as "3 is less than 5" and "-5 is less than -3"

Greater than:

$>$, when read from left to right means greater than.

Eg. $5 > 3$ or $-3 > -5$

Described as "5 is greater than 3" and "negative 3 is greater than negative 5"

Examples: Place the greater-than or less-than symbols to make the statements true.

a) $2 < 8$ b) $-7 < 3$

Some collections of fractions are easier to differentiate than others. Obviously we can see that ¼ is less than ¾ or than ⅜ is bigger than $\frac{3}{11}$, but what about $\frac{2}{7}$ and ⅜ or $\frac{16}{47}$ and $\frac{19}{51}$? We will observe two methods to solve such problems.

Option 1: Turn them into decimals.

Note: This tactic is the fastest, but will often require the use of a calculator.

Method: Divide the numerator by the denominator for both fractions and compare them by traditional methods.

Example: Place the inequality appropriately.

$\frac{2}{7} < \frac{3}{8}$ $\frac{5}{11} > \frac{4}{9}$

$.28571\ldots < .375$ $.\overline{45} > .\overline{44}$

Option 2: Convert each fraction so they have a common denominator.
Note: You will use this method more often in our class as it does not require the use of a calculator.

Method: Determine a common denominator between the two fractions by multiplying both fractions by a version of 1. Compare the numerators.

Example: Place the inequality appropriately.

$\frac{2}{7} < \frac{3}{8}$ $\frac{5}{11} > \frac{4}{9}$

$\left(\frac{8}{8}\right)\frac{2}{7}$ $\frac{3}{8}\left(\frac{7}{7}\right)$

$\frac{16}{56} < \frac{21}{56}$

$\left(\frac{9}{9}\right)\frac{5}{11}$ $\frac{4}{9}\left(\frac{11}{11}\right)$

$\frac{45}{99} > \frac{44}{99}$

3.2 - Adding and Subtracting Rational Numbers

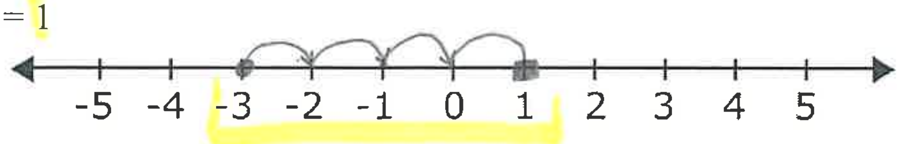
Number lines are an excellent way of visualizing addition and subtraction. In general, when adding two numbers $a + b$, we start at the location of a, and move according to b.

- If b is positive, move from a to the right.
- If b is negative, move from a to the left.
- If b is zero, stay at a.

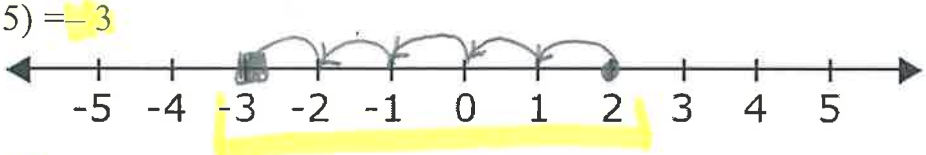
We will begin our investigation of operating on rational numbers with terminating values. In the next section, we will observe how to combine fractions.

Examples: Using a number line, find,

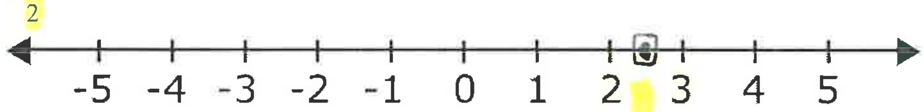
a) $-3+4=1$



b) $2+(-5)=-3$



c) $\frac{5}{2}+0=\frac{5}{2}$



Rules for Subtracting Rational Numbers

For any rational number a and b: $a+(-b)=a-b$

The rules for subtraction are the same as the rules for addition.

Therefore the following statements, while written differently, are equal to one another.

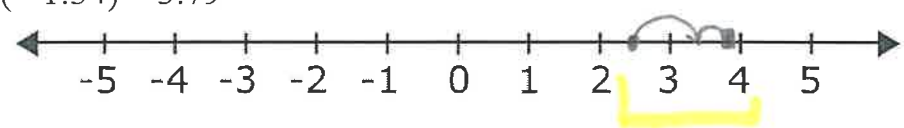
Adding an opposite	Subtraction
$3+(-8)=-5$	$3-8=-5$
$-4+(-2)=-6$	$-4-2=-6$
$-5+9=4$	$-5-(-9)=4$

Adding or Subtracting Decimals

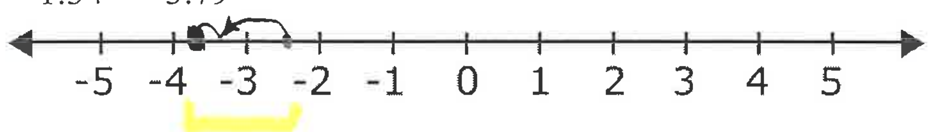
The rules for adding and subtracting decimals follow the same rules as the integers -- just make sure you use correct place-value comprehension.

Examples:

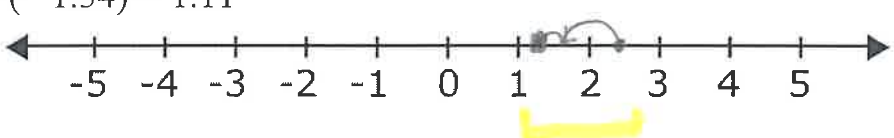
d) $2.45 - (-1.34) = 3.79$



e) $-2.45 - 1.34 = -3.79$



f) $2.45 + (-1.34) = 1.11$



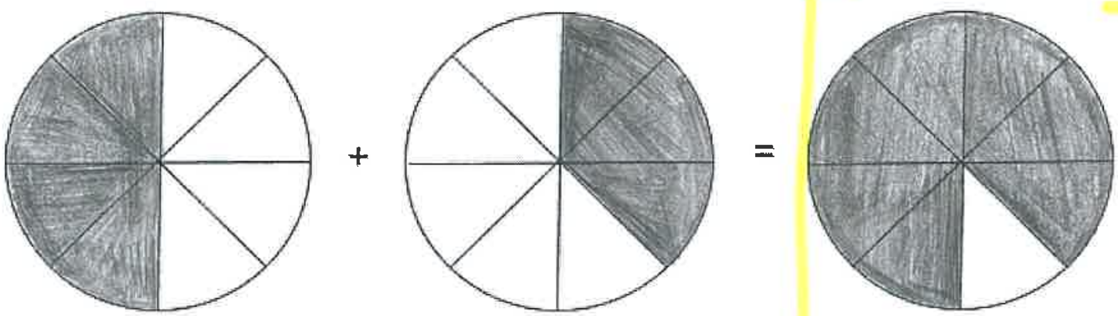
3.3 - Adding and Subtracting Using Fraction Notation

When adding and subtracting fractions we must take note of the denominators. We will investigate operating on fraction where the denominators are the same (known as like denominators) or if they are different (known as unlike denominators).

Adding and subtracting fractions with like denominators

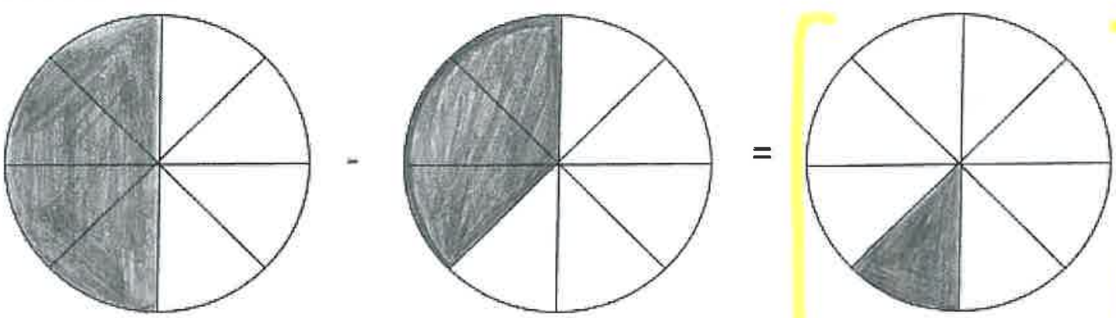
Observe:

Addition



$$\frac{4}{8} + \frac{3}{8} = \frac{7}{8}$$

Subtraction



$$\frac{4}{8} - \frac{3}{8} = \frac{1}{8}$$

We can see that for adding and subtracting fractions that have a common denominator, it is as simple as:

1) Add/Subtract the numerators.
2) Keep the denominators the same (do not combine them).
3) Simplify if possible.

Examples:

a) $\frac{3}{2}+\frac{1}{2}=$ $\frac{4}{2}=2$

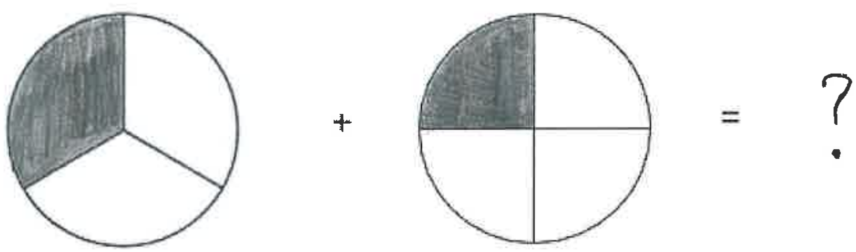
b) $\frac{3}{2}-\frac{1}{2}=$ $\frac{2}{2}=1$

c) $\frac{12}{5}+\frac{4}{5}=$ $\frac{16}{5}$

d) $\frac{12}{5}+(-\frac{4}{5})=$ $\frac{12}{5}-\frac{4}{5}=\frac{8}{5}$

<u>Adding and subtracting fractions with unlike denominators</u>

When adding or subtracting with unlike denominators, we run into the issue of having different sized pieces being combined together. Take the following visual example of $\frac{1}{3}+\frac{1}{4}$.



We can see that one can not simply just add the numerators together as the two slices are not of equal size; also, what would the denominator be -- thirds or quarters?

When adding or subtracting fractions we must make sure the denominators are the same! When they are not, we must adjust them (by multiplying by versions of 1) so that they become the same.

Let's investigate how to adjust our fractions.

We already know that anything multiplied by 1 is itself. We also have seen that the number 1 can be written as a fraction -- when the numerator and denominator are the same. $1=\frac{1}{1}=\frac{5}{5}=\frac{-3}{-3}$

So,

$\frac{1}{2}(1) = \frac{1}{2}$

$\frac{1}{2}(\frac{2}{2}) = \frac{2}{4} = \frac{1}{2}$

$\frac{1}{2}(\frac{5}{5}) = \frac{5}{10} = \frac{1}{2}$

$\frac{1}{2}(\frac{-3}{-3}) = \frac{-3}{-6} = \frac{1}{2}$

We have now observed two important things in math.

1) We can manipulate how fractions look while keeping their value the same by multiplying them by versions of 1.
2) When you multiply a fraction, simply multiply the numerators together to form the new numerator-product and then multiply the denominators together to from the denominator-product. (We will investigate this officially in the next section).

Okay, back to our original fractions with unlike denominators.

$\frac{1}{3} + \frac{1}{4} =$

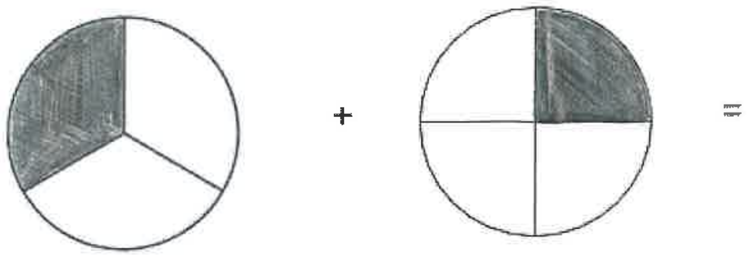
Using some number-sense, we can determine that both our denominators of 3 and 4 can perfectly factor 12. We will thus multiply each fraction accordingly so that they still hold their original values but are both out of 12.

$\frac{1}{3}(\frac{4}{4}) + \frac{1}{4}(\frac{3}{3}) = \frac{1\times4}{3\times4} + \frac{1\times3}{4\times3} = \frac{4}{12} + \frac{3}{12}$

Now that the fractions have the same denominators, we can simply add them as we did at the beginning of the section.

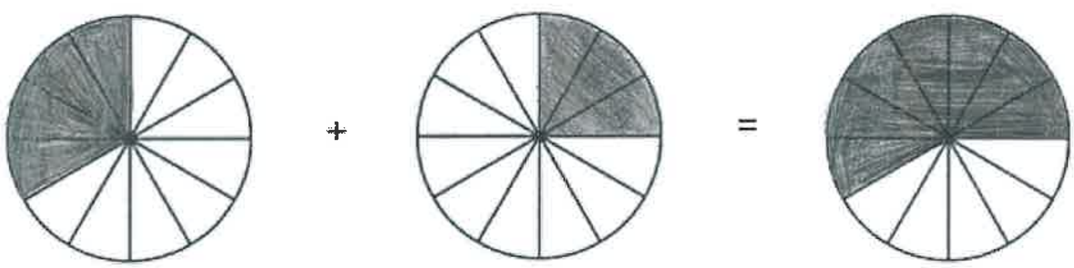
$\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

Visually, this is what we just did:



$\frac{1}{3}$ + $\frac{1}{4}$

$\frac{1}{3}\left(\frac{4}{4}\right)$ + $\frac{1}{4}\left(\frac{3}{3}\right)$ = $\frac{}{12}$



$\frac{4}{12}$ + $\frac{3}{12}$ = $\frac{7}{12}$

Examples: Simplify the following

a) $\frac{3}{8}+\frac{1}{4}$

$= \frac{3}{8}+\frac{1}{4}\left(\frac{2}{2}\right) = \frac{3}{8}+\frac{2}{8}$

$= \boxed{\frac{5}{8}}$

b) $-\frac{5}{7}+\frac{11}{14}$

$= \left(\frac{2}{2}\right)\frac{-5}{7}+\frac{11}{14} = \frac{-10}{14}+\frac{11}{14} = \boxed{\frac{1}{14}}$

c) $\frac{1}{3}+\frac{5}{4}$

$= \left(\frac{4}{4}\right)\frac{1}{3}+\frac{5}{4}\left(\frac{3}{3}\right) = \frac{4}{12}+\frac{15}{12}$

$= \boxed{\frac{19}{12}}$

d) $\frac{11}{5}-\frac{1}{6}$

$= \left(\frac{6}{6}\right)\frac{11}{5}-\frac{1}{6}\left(\frac{5}{5}\right) = \frac{66}{30}-\frac{5}{30} = \boxed{\frac{61}{30}}$

Converting Mixed Fractions and Improper Fractions

The fractions we are used to seeing have a denominator that is larger than the numerator (½ and ¾ for example), often however the numerator may the larger than the denominator (like $\frac{2}{1}$ and $\frac{4}{3}$). These fractions (where the numerator is larger than the denominator) are known as "improper fractions." Improper fractions are often rewritten as "mixed fractions" which denote how many full times the denominator fits into the numerator. Instead of seeing $\frac{3}{2}$, read as "three halves," we usually see $1\frac{1}{2}$, read as "one and a half."

It is important to be able to convert between the two types for a couple of reasons:

1) It is easier to operate on fractions that are in the improper form, not mixed.
2) You will need to present your answer as it is required by the posed question.

Let's observe how to convert between the two.

<u>Converting from Mixed Fractions to Improper Fractions</u>

1) Multiply the whole number by the denominator
2) Add the product to the numerator (this new sum becomes the numerator)
3) Keep the original denominator (this value is the denominator)

Note about negatives: If it is a negative value put the negative sign aside, follow steps 1-3, then place the negative sign back in at the end.

Example: Convert the following from mixed fractions into improper fractions.

a) $3\frac{2}{5}$

$= 3\frac{2}{5}$ + ×

$= \frac{3 \times 5 + 2}{5}$

$= \boxed{\frac{17}{5}}$

b) $-5\frac{1}{4}$

$= -5\frac{1}{4}$ + ×

$= -\frac{(5 \times 4 + 1)}{4}$

$= -\frac{(21)}{4} = \boxed{\frac{-21}{4}}$

Converting from Improper Fractions to Mixed Fractions

1) Divide the denominator into the numerator to produce the whole number.
   IOW: *How many times does the denominator fully go into the numerator?*
2) Multiply the found whole number by the denominator, and subtract this product from the original numerator. This will produce your new numerator.
   IOW: *What is left over?*
3) Keep the denominator.

Note about negatives: If it is a negative value put the negative sign aside, follow steps 1-3, then place the negative sign back in at the end.

Example: Convert the following from improper fractions into mixed fractions.

a) $\frac{12}{5}$ • $5 \rightarrow 12$: twice ← whole
- with 2 left over ← num
- originally out of 5 ← den

$\frac{12}{5} = \boxed{2\,^2/_5}$

b) $\frac{-26}{4}$ • $4 \rightarrow 26$: 6 times ← whole
- with 2 left over ← num
- originally out of 4 ← den

$\frac{-26}{4} = 6\,^2/_4 = \boxed{6\,^1/_2}$

Note about denominators: As you can see when converting between the two types of fractions we leave the denominator the same. The denominator tells us how the fraction is split up, which does not change when we convert between fraction types.

The Least Common Multiple (LCM) of a Set of Numbers

The lowest common multiple (LCM) is the smallest number that two (or more) natural numbers perfectly factor.

What is the LCM of:

a) 4 and 3

Answer: LCM = 12. Both numbers also go into 24 and 36 and 1200, but 12 is the lowest.

b) 6 and 18

Answer: LCM = 18. 6 goes into 18 three times. 18 goes into 18 one full time.

c) 18 and 90

These can be more challenging. We will investigate this next.

To fully learn about the LCM we must discuss prime numbers.

Prime Numbers and Prime Factor Decomposition

Prime numbers are the building blocks of all numbers. They are numbers that can only be divided by one and itself.

Eg. 2, 3, 5, 7, 11, 13, 17, 19…

Important facts about primes:

- 0 and 1 are not prime numbers; they are exceptions to our prime system.
- There are an infinite number of primes. The biggest known prime is $2^{74207281}-1$ which was discovered on January 4, 2018; it has 23,249,425 digits!
- Every real number is either itself a prime or can be reached by multiplying primes together (these numbers are known as composites):

Eg. **1** (exception), **2** (prime), **3** (prime), **4** (2×2), **5** (prime), **6** (2×3), **7** (prime), **8** ($2 \times 2 \times 2$), **9** (3×3), **10** (5×2), **11** (prime), **12** ($4 \times 3 = 2 \times 2 \times 3$)…

That fact is essential to our number system as it means we can decompose every number into a product of its primes. Once we do that, finding the LCM is quite simple actually.

Finding the LCM using a factor tree

1) Write each number as a product of prime factors
2) Select all the unique primes and their greatest powers
3) Multiply the selection together

Determine the LCM of:

c) 18 and 90

18
2 9
3 3

90
10 9
5 2 3 3

$18: 2^1 \times 3^2$ $90: 2^1 \times 3^2 \times 5^1$

LCM: $2^{\square} \times 3^{\square} \times 5^{\square}$

$= 2^1 \times 3^2 \times 5^1$

$= 2 \times 9 \times 5$

$= \boxed{90}$

d) 24 and 60

24
6 4
3 2 2 2

60
6 10
2 3 5 2

$24: 2^3 \times 3^1$ $60: 2^2 \times 3^1 \times 5^1$

LCM: $2^{\square} \times 3^{\square} \times 5^{\square}$

$= 2^3 \times 3^1 \times 5^1$

$= 8 \times 3 \times 5$

$= \boxed{120}$

Solve: $3\frac{3}{8} + 2\frac{5}{18}$

① 8
2 4
2 2

18
9 2
3 3

$8: 2^3$ $18: 2^1 \times 3^2$

② LCM

$2^3 \times 3^2$

$= 8 \times 9$

$= \underline{72}$

③ $3\frac{3}{8}\left(\frac{9}{9}\right) + 2\frac{5}{18}\left(\frac{4}{4}\right)$

$= 3\frac{27}{72} + 2\frac{20}{72}$

$= \boxed{5\frac{47}{72}}$

3.4 - Multiplying and Dividing Rational Numbers

Thankfully, multiplying and dividing rational numbers is simpler than adding and subtracting them.

Multiplying Fractions

Multiplying Fractions by Whole Numbers

Recall that multiplication is the short form for values that are repeatedly added together. For example, $2 \times 4 = 2 + 2 + 2 + 2$.
We can extend this principal to fractions:
For example $\frac{2}{3} \times 5 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$ and because all of these values have the same denominator, we can keep the denominator as is and simply add up all the numerators.
Therefore: $\frac{2}{3} \times 5 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{10}{3}$

Multiplying Two Fractions together

*A visual representation of multiplying fractions is provided at the back of the booklet.

Rule: When multiplying fractions, exclusively multiply the numerators together (to form your new numerator), then multiply your denominators together (to form your new denominator).

Examples:

a) $\frac{3}{2} \times \frac{5}{11} = \frac{3 \times 5}{2 \times 11} = \frac{15}{22}$

b) $\frac{4}{5} \times \frac{-9}{2} = \frac{4 \times -9}{5 \times 2} = \frac{-36}{10} = \frac{-18}{5}$

Note: This holds true for multiplying fractions by integers too. Simply turn the integer into a fraction (by putting it over 1) and proceed.

Example:

c) $\frac{5}{6} \times 4 = \frac{5}{6} \times \frac{4}{1} = \frac{5 \times 4}{6 \times 1} = \frac{20}{6} = \frac{10}{3}$

Dividing Fractions

*A visual representation of dividing fractions is provided at the back of the booklet.

Rule: When dividing fractions, follow the steps of "keep-it, change-it, flip-it."

Keep the first fraction as is.
Change the division symbol to a multiplication symbol
Flip the second fraction over to its reciprocal.

Examples:

a) $\frac{3}{2} \div \frac{5}{11} = \frac{3}{2} \times \frac{11}{5} = \frac{33}{10}$

b) $\frac{4}{5} \div \frac{-9}{2} = \frac{4}{5} \times \frac{2}{-9} = \frac{8}{-45} = \frac{-8}{45}$

Note: This holds true for dividing fractions by integers too. Simply turn the integer into a fraction (by putting it over 1) and proceed.

Example:
c) $\frac{5}{6} \div -4 = \frac{5}{6} \div \frac{-4}{1} = \frac{5}{6} \times \frac{1}{-4} = \frac{5}{-24} = \frac{-5}{24}$

Simplifying Fractions

When multiplying or dividing fractions, we almost always need to simplify our solution so it is in "lowest terms." This is traditionally done as your last step. However, the quickest and easiest way to simplify is actually to reduce your fractions (if possible) before multiplying or dividing. This is done by identifying and "cancelling" any common factors that appear in both the numerator and the denominator.

For example:

32 → 2, 16 → 2, 8 → 2, 4 → 2, 2

$32 = 2^5$

a) $\frac{25}{32} \times \frac{12}{32} = \frac{5\times5}{2\times2\times2\times2\times2} \times \frac{\cancel{2}\times\cancel{2}\times3}{\cancel{2}\times\cancel{2}\times2\times2\times2}$

$= \frac{5\times5\times3}{2^8} = \boxed{\frac{75}{256}}$

b) $\frac{9}{16} \div \frac{21}{10} = \frac{9}{16} \times \frac{10}{21} = \frac{\cancel{3}\times3}{\cancel{2}\times2\times2\times2} \times \frac{5\times\cancel{2}}{7\times\cancel{3}}$

$= \frac{3\times5}{2\times2\times2\times7} = \boxed{\frac{15}{56}}$

Multiplying and Dividing with Mixed Fractions and Decimals

When multiplying or dividing with mixed fractions and decimals you must convert your values into improper fractions before you can find your solution.

Examples: must convert to improper fractions first

a) $3\frac{2}{5} \div 2\frac{1}{3}$

$= \frac{17}{5} \div \frac{7}{3}$

$= \frac{17}{5} \times \frac{3}{7}$

$= \boxed{\frac{51}{35}} = \boxed{1\frac{16}{35}}$

b) $5\frac{1}{2} \times 2.6 = 5\frac{1}{2} \times 2\frac{6}{10}$

$= \frac{11}{2} \times \frac{26}{10} = \frac{11}{2} \times \frac{13\times\cancel{2}}{5\times\cancel{2}}$

$= \frac{11}{2} \times \frac{13}{5} = \boxed{\frac{143}{10}} = \boxed{14\frac{3}{10}}$

Notes:

Fractions

$\frac{1}{1} = 1$

$\frac{1}{2} = .5$

$\frac{1}{3} = .\overline{3}$

$\frac{1}{4} = .25$

$\frac{1}{5} = .2$

$\frac{1}{6} = .1\overline{6}$

$\frac{1}{7} \cong .14$

$\frac{1}{8} = .125$

$\frac{1}{9} = .\overline{1}$

$\frac{1}{10} = .1$