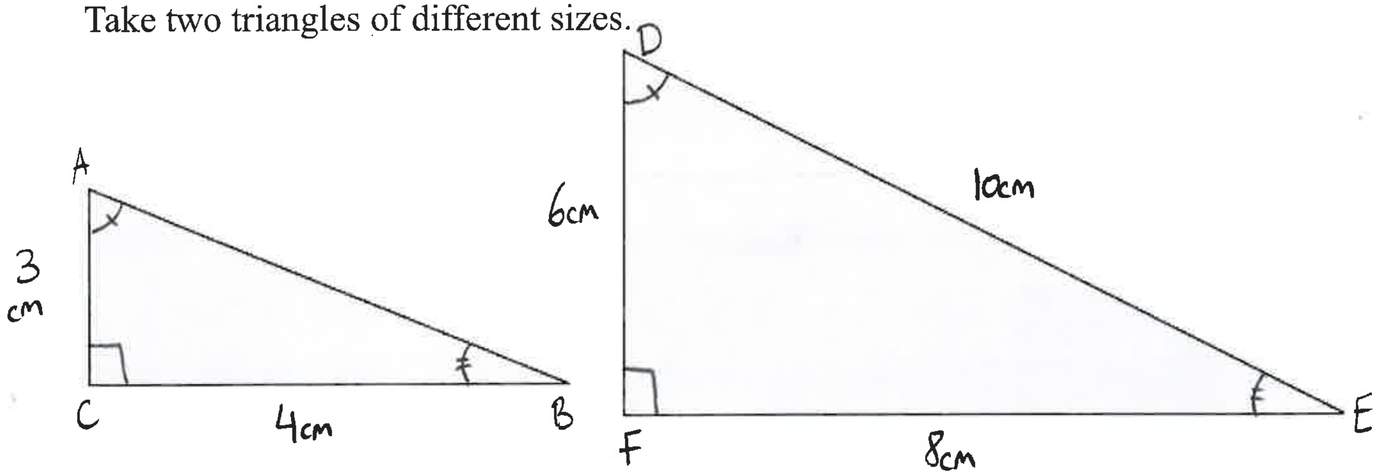
Chapter 9 - Similarity and Scale Factor

9.1 - Scale Factor

One of the most underappreciated mathematical applications is the concept of similarity and scale factors. We use it all the time, in road maps, astronomy, trigonometry, and beyond. Scale factors allow us to imagine full-sized objects in a more manageable setting.

Similar Figures:

Take two triangles of different sizes.

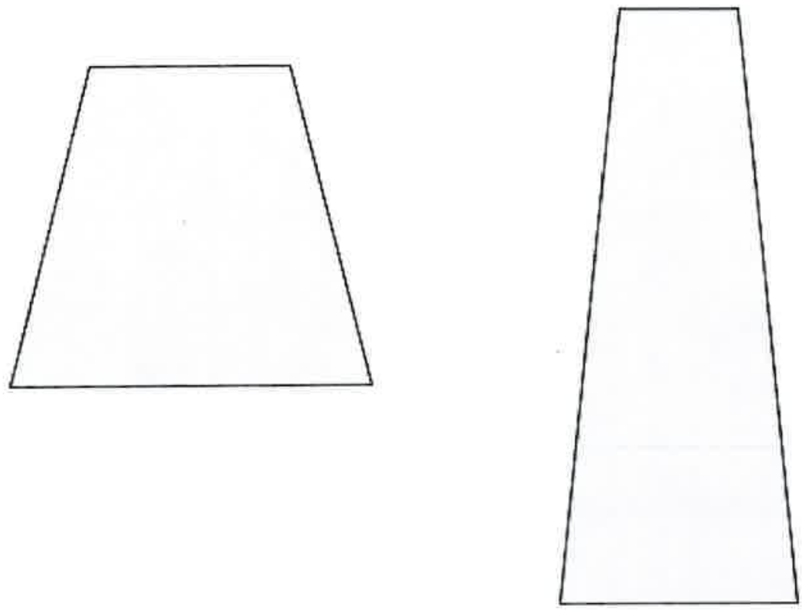


The figures may only look alike, but they are also related to one another mathematically. Furthermore, information about one of these triangles can provide us with enough data to solve problems about the other triangle. For instance, we can see that if the legs of the larger triangle are twice the size of that of the smaller triangle (this is known as the scale factor) then so too must the hypotenuse be twice the size.

So what is the length of the hypotenuse of the smaller triangle?

Big triangle is twice the size, so hyp of small = $\boxed{5cm}$

Now obviously not every two objects are related in this way. Take the following polygons:



While these two objects are both trapezoids, they are not mathematically similar to one another. The data from one, can not be transferred to tell us anything significant about the other.

For two objects to be mathematically "similar" three facts must be true:

a) The two objects must be fundamentally the same shape
b) The corresponding angles are the same
c) The corresponding lengths must be proportional.

The symbol ~ indicates that figures are similar to one another.
For example for our above triangles, we would say that $\Delta ABC \sim \Delta DEF$ with a scale factor of 2.

Scale Factor

If two objects are similar to one another, we then care about their relative scale factor (the magnitude which states the proportional relationship between the corresponding shapes). To find this value we must first name the two shapes -- original is known as the "object" whereas it's similar shape is known as the "image."

FORMULA: $Scale\ factor = \frac{image\ length}{object\ length}$

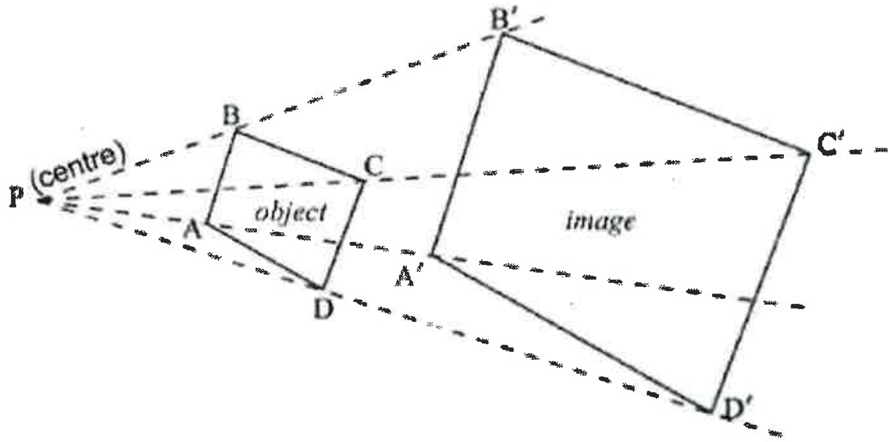
- If the image is the same size as the object (like your face in a mirror, or drawing a perfect life-sized painting of an apple) we say the scale factor is one.
- If the image is larger than the object (like when making drawings of ants or atoms), the scale factor is greater than one. This is known as an *enlargement.*
- If the image is smaller than the object (like when drawing a map, or making a model of a plane), the scale factor is less than one. This is known as a *reduction.*

Notes:

1) PA is the distance between point P and point A; BC would thus be the distance between point B and point C. It is not to be confused with $P \times A$ and $B \times C$.

2) While the object uses letters ABCD for each vertex, the image uses A'B'C'D' (read as "A-prime, B-prime, C-prime, and D-prime") to denote the corresponding points. It is vitally important to label your drawings correctly, be careful when labelling as some images may be rotated or flipped from the original objects. There are two ways of finding the scale factor of similar shapes.

Consider the following diagram:



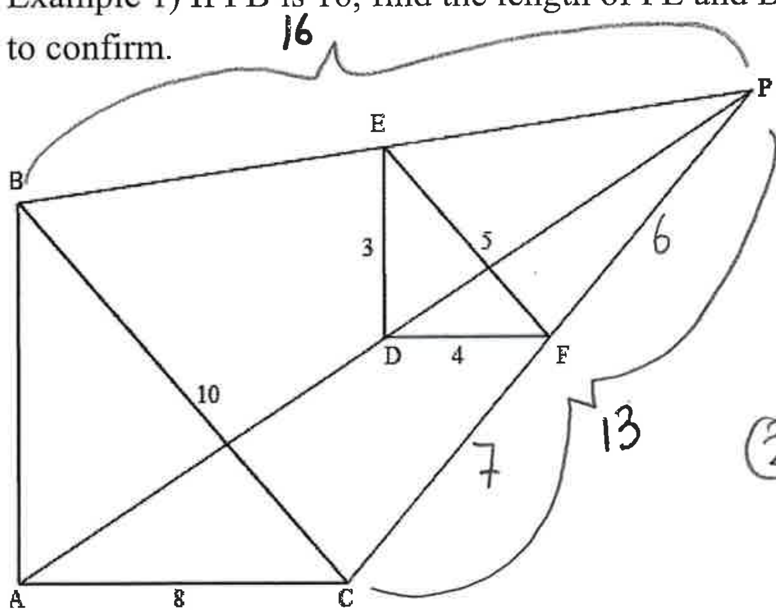
Method 1 (most common): The measure between corresponding vertices.

$$\text{Scale Factor} = \frac{\textit{Image length}}{\textit{Object length}} = \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA}$$

Method 2 (less common): The measure between the centre point of each object and their corresponding vertices. Let P be the centre between two similar shapes.

$$\text{Scale Factor} = \frac{\textit{Image length}}{\textit{Object length}} = \frac{PA'}{PA} = \frac{PB'}{PB} = \frac{PC'}{PC} = \frac{PD'}{PD}$$

Example 1) If PB is 16, find the length of PE and BA. Use the scale factor formula to confirm.



① $\frac{PE}{PB} = \frac{PF}{PC} \Rightarrow \frac{PE}{16} = \frac{6}{13}$

$PE = \left(\frac{6}{13}\right)16$

$\boxed{PE = 7.38}$

② $\frac{BA}{ED} = \frac{BC}{EF} \Rightarrow \frac{BA}{3} = \frac{10}{5}$

$BA = \frac{3 \times 10}{5}$

$\boxed{BA = 6}$

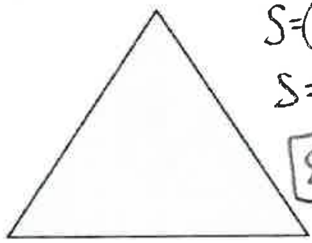
The sum of interior angles

No matter the size of a shape, or even what the shape looks like, the total sum of all of the interior angles is a standard value solely dependent on the number of sides the polygon has. That means that every possible triangle has a total interior-angle-sum of 180° and every sized trapezoid will sum to 360°. This relationship can be defined by the following formula (where S is the interior angle sum, and n is the number of sides):

FORMULA: $S = (n - 2) \times 180^\circ$

Example 2) Find the interior-angle-sum and determine the value of the missing angle.

a)

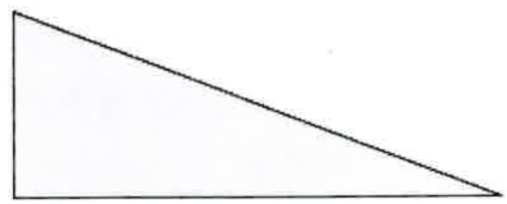


$S = (3-2) \times 180$

$S = 1(180)$

$\boxed{S = 180^\circ}$

b)

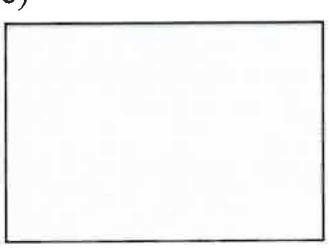


$S = (3-2) \times 180$

$S = 1 \times 180$

$\boxed{S = 180^\circ}$

c)

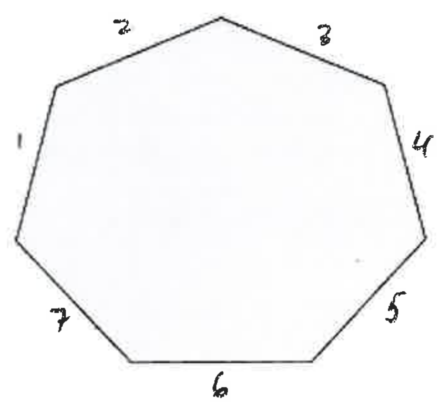


$S = (4-2) \times 180$

$S = 2 \times 180$

$\boxed{S = 360^\circ}$

d)



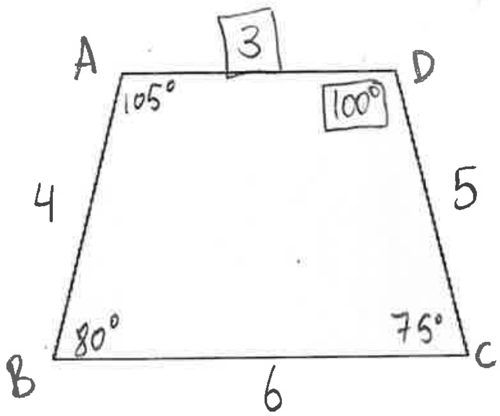
$S = (7-2) \times 180$

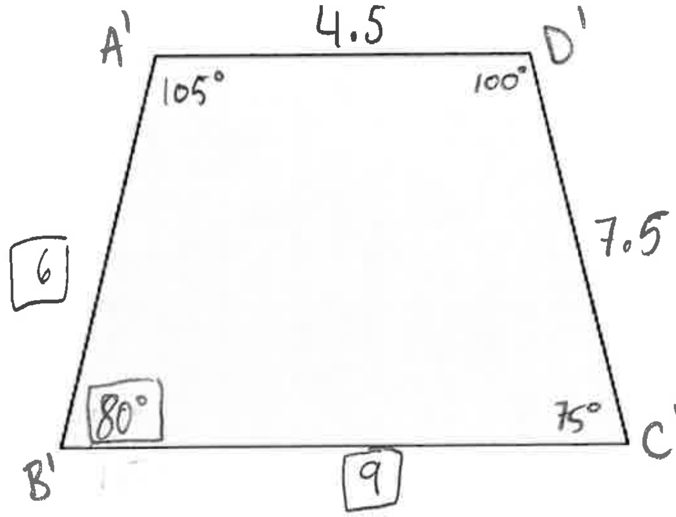
$S = 5 \times 180$

$\boxed{S = 900^\circ}$

Example 3: Determine if the following shapes are similar. If they are, determine the scale factor and solve for all unknown angles and sides.

a)





Sum of interior angles: $(n-2)\times 180 = (4-2)\times 180 = 360°$

object: $360-(105+75+80) = 100°$ Image: $360-(105+75+100) = 80$

$\therefore ABCD \sim A'B'C'D'$

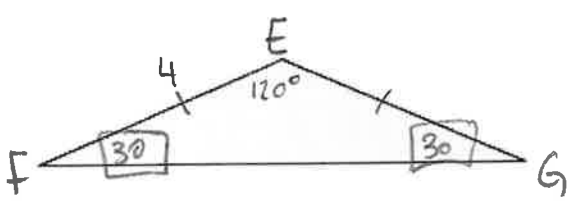
Scale Factor: $\frac{D'C'}{DC} = \frac{7.5}{5} = \boxed{1.5}$

$BC = 6$ So $B'C' = 6 \times 1.5 = \boxed{9}$

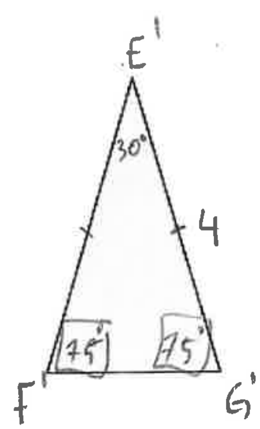
$AB = 4$ So $A'B' = 4 \times 1.5 = \boxed{6}$

$A'D' = 4.5 \Rightarrow 1.5 = \frac{4.5}{AD}$ $AD = \frac{4.5}{1.5}$ $\boxed{AD = 3}$

b)



$180 - 120 = 60° \div 2 = 30°$



$180 - 30 = 150 \div 2 = 75°$

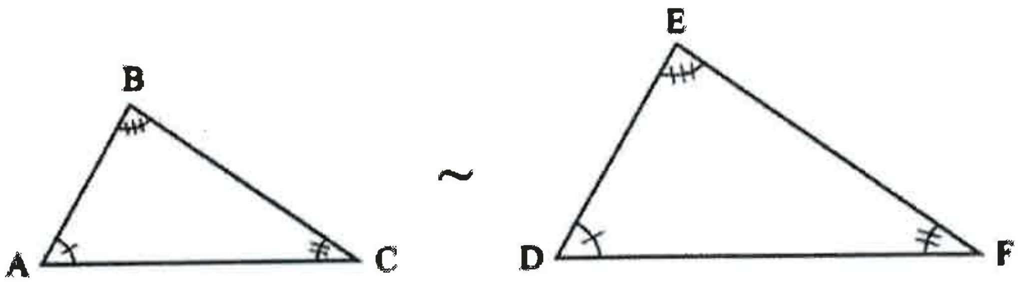
$\therefore$ Not Similar!

9.2 - Similar Triangles

Recall that two polygons are similar if:

a) The corresponding angles are equal
b) The corresponding sides are proportional

For the following two triangles:



1) $\angle A = \angle D$ (May also be defined as $\angle BAC = \angle EDF$)

2) $\angle B = \angle E$ ($\angle ABC = \angle FED$)

3) $\angle C = \angle F$ ($\angle ACB = \angle DFE$)

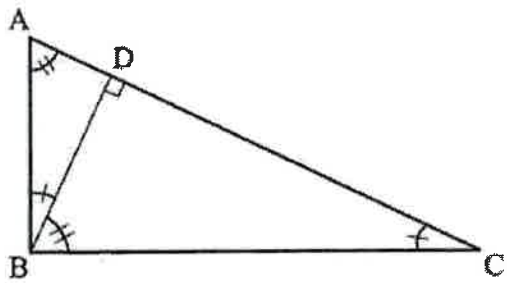
Therefore, the two triangles are similar, and thus have proportional corresponding side lengths. This can be written mathematically:

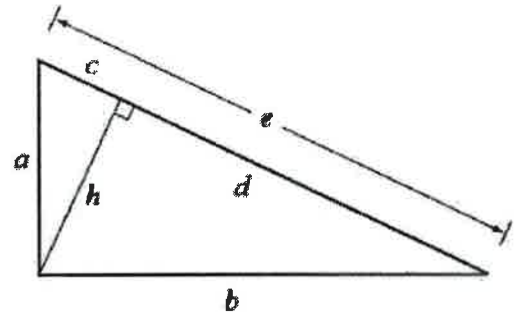
$\Delta ABC \sim \Delta DEF$, thus $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

Similarity Properties in Right Triangles

<u>The Altitude Property</u>:

The altitude to the hypotenuse (a line that forms a right angle with the hypotenuse, and originates at the right-angle vertex) of a right triangle, forms two new triangles that are similar to each other and to the original triangle.

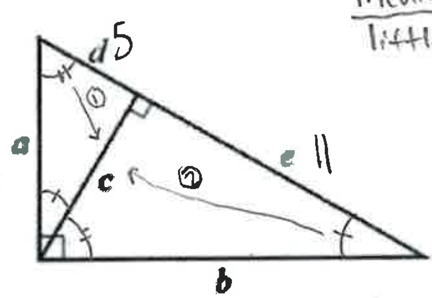




$\Delta ABC \sim \Delta BDC \sim \Delta ABD$

$$\frac{c}{h} = \frac{h}{d} = \frac{a}{b}, \quad \frac{e}{a} = \frac{a}{c} = \frac{b}{h}, \quad \frac{e}{b} = \frac{b}{d} = \frac{a}{h}$$

Example: If $d = 5$, $e = 11$, find c.



$$\frac{\text{medium}}{\text{little}} = \frac{11}{c} = \frac{c}{5}$$

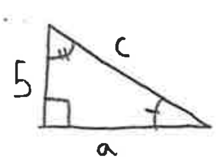
$$55 = c^2$$

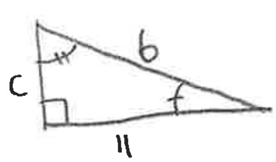
$$c = \sqrt{55} \sim 7.4$$

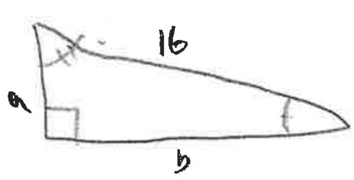
$$\overbrace{\sqrt{49} \quad \sqrt{55}}^{6} \quad \sqrt{64}$$

$$\underbrace{\sqrt{49} \quad \sqrt{55} \quad \sqrt{64}}_{15}$$

$$7\tfrac{6}{15} \sim 7.4$$



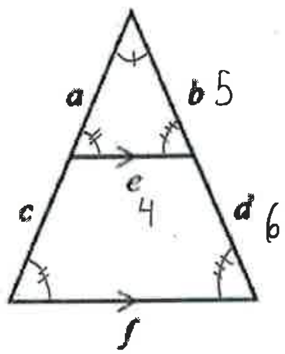




The Parallel Property

If a line that runs parallel to a leg of a triangle is drawn through two sides of a triangle, it forms a similar triangle.

Example: If $e = 4$, $b = 5$, and d = 6, find f.



$$\frac{e}{f} = \frac{b}{bd}$$

$$\frac{4}{f} = \frac{5}{11}$$

$$\frac{f}{e} = \frac{bd}{b}$$

$$\frac{f}{4} = \frac{11}{5}$$

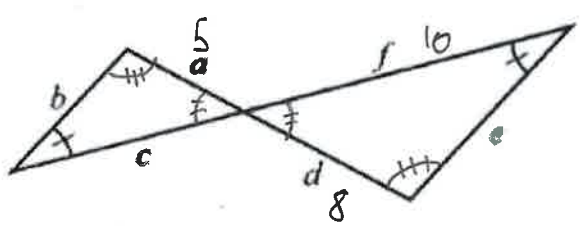
$$f = \frac{44}{5}$$

$$\boxed{f = 8.8}$$

The X-Angle Property

Two intersecting lines form four angles. The angles opposing one another are equal and the sum of the four angles equals 360°.

Example: If $a = 5$, $d = 8$, $f = 10$, find c.



$$\frac{c}{a} = \frac{f}{d}$$

$$\frac{c}{5} = \frac{10}{8}$$

$$c = \frac{50}{8}$$

$$\boxed{c = 6.25}$$

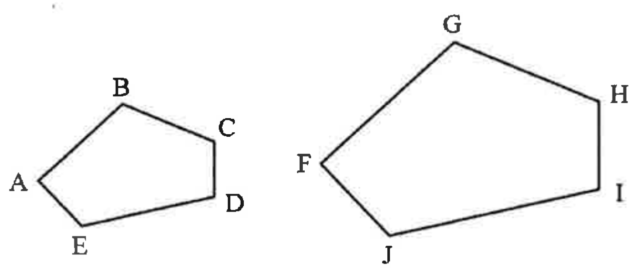
9.3 - Similar Polygons

Recall:

1) Two shapes are similar if the corresponding angles are equal sides are proportional.
2) The sum of the interior angles can be defined by $S = 180(n-2)$

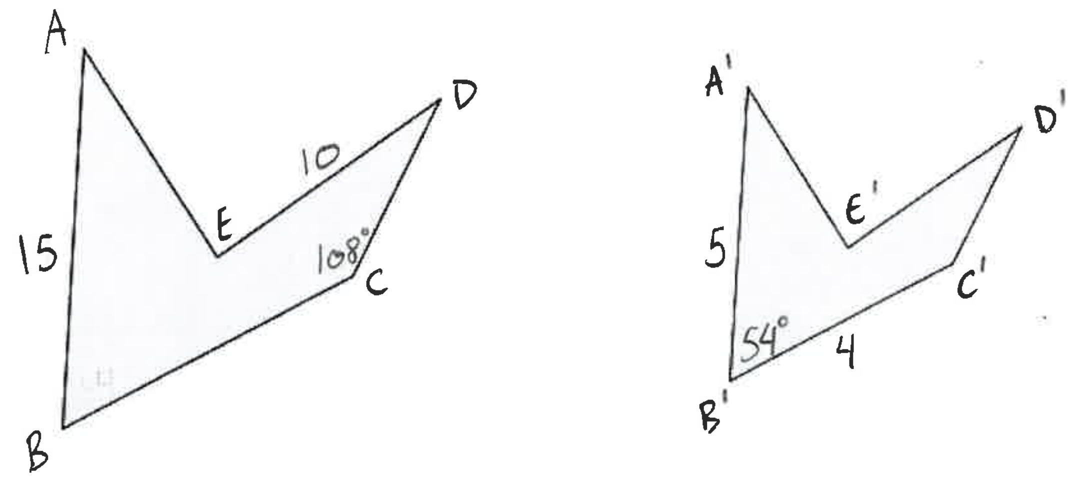
We can fully apply what we have discussed about triangles to that of polygons.

If Polygons ABCDE and FGHIJ are similar, then:



Therefore $\angle A = \angle F$, $\angle B = \angle G$, $\angle C = \angle H$, $\angle D = \angle I$, $\angle E = \angle J$ and $\frac{AB}{FG} = \frac{BC}{GH} = \frac{CD}{HI} = \frac{DE}{IJ} = \frac{EA}{JF}$

Example 1) $ABCDE \sim A'B'C'D'E'$.



Find a) $\angle B$
b) $\angle C'$
c) CB
d) $D'E'$
e) the scale factor

a) $\angle B = 54°$

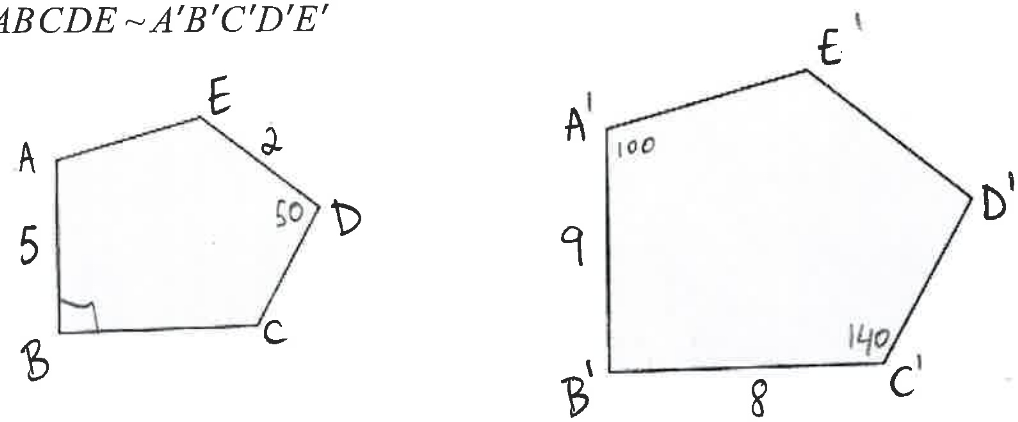
b) $\angle C' = 108°$

c) $\frac{CB}{C'B'} = \frac{AB}{A'B'} \Rightarrow \frac{CB}{4} = \frac{15}{5} \Rightarrow CB = \frac{60}{5} \Rightarrow \boxed{CB = 12}$

d) $\frac{D'E'}{DE} = \frac{A'B'}{AB} \Rightarrow \frac{D'E'}{10} = \frac{5}{15} \Rightarrow \frac{50}{15} = \boxed{D'E' = 3.\overline{3}}$

e) $\frac{\text{image}}{\text{object}} = \frac{5}{15} = \boxed{\frac{1}{3}}$ = reduction

Example 2) $ABCDE \sim A'B'C'D'E'$



Find a) $\angle A$

b) $\angle B'$

c) CB

d) $D'E'$

e) the scale factor

a) $\angle A = 100°$

b) $\angle B' = 90°$

c) $\frac{CB}{C'B'} = \frac{AB}{A'B'} \Rightarrow \frac{CB}{8} = \frac{5}{9} \Rightarrow CB = \frac{40}{9} = \boxed{4.\overline{4}}$

d) $\frac{D'E'}{DE} = \frac{A'B'}{AB} \Rightarrow \frac{D'E'}{2} = \frac{9}{5} = D'E' = \frac{18}{5} = \boxed{3.6}$

e) $\frac{\text{Image}}{\text{object}} = \frac{9}{5} = \boxed{1.8x}$ = enlargement

Notes: