PC12 Name____________________________

# Polynomials

# Chapter Notes

Key

## Assignment List

| Date | Lesson | Assignment |
|---|---|---|
| | 0. Polynomials Exploration | Polynomials Exploration Handout |
| | 1. Features of Polynomials I | Mickelson Page 119 #2-3, 6a-f, 7acegi & Page 127 #1 |
| | 2. Features of Polynomials II | Mickelson Page 120 #4, 9abcehijm (for 9hij use factoring by grouping) #5 |
| | 3. Graphing Polynomials | Mickelson Page 127 #2ab, 3ab, 5ab, 6, 7ab & Graphing Polynomials Worksheet |
| | 4. Polynomial Division | Mickelson Page 135 #1ac, 2acfhk, 3bcdgh, 4a-d |
| | 5. Remainder and Factor Theorem | Mickelson Page 143 #1, 2abcd, 3abce |
| | 6. Rational Root Theorem | Mickelson Page 144 #4a-e, 5ace and Polynomials Supplement |
| | | Practice Test |
| | | Review |
| | | **Polynomials Test** |

# Polynomials Day 1: Features of Polynomials I

**Key Vocabulary:**

| | | | |
|---|---|---|---|
| **Polynomial** | **Leading Coefficient** | **Zeros** | **Multiplicity** |
| **Leading Term** | **End behaviour** | **x-intercepts** | |
| **Degree** | **Turning Points** | **y-intercept** | |

A polynomial expression has the form:

$$ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots + ix^2 + jx + k, \quad where\ a \neq 0 \text{ and}$$

1. exponents must be whole numbers (may skip descending powers)
2. coefficients must be real numbers

The polynomial's degree is defined by $n$ and its leading coefficient is $a$.

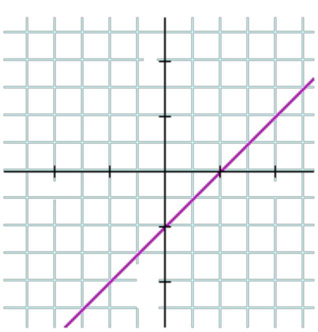
↑highest power of $x$ ↑coeff. of highest power of $x$

**Example 1:**

| Polynomial | Degree | Leading coefficient | Name |
|---|---|---|---|
| $f(x) = 5x^4 - 4x^2 - 3$ | 4 | 5 | Polynomial |
| $g(x) = \sqrt{5}x^3 - 2x^2 - 3x$ | 3 | $\sqrt{5}$ | Cubic |
| $h(x) = -3x^2 - 3$ | 2 | $-3$ | Quadratic |
| $p(x) = -3 + 4x^1$ | 1 | $+4$ | Linear |
| $d(x) = -3x^0$ | 0 | $-3$ | Constant |

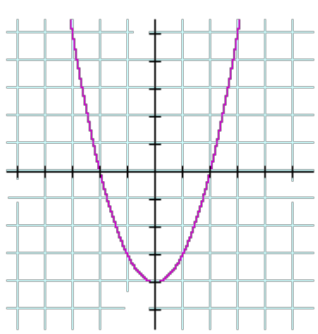
**Shape:** Polynomial functions are continuous, with no breaks or sharp corners.

Are these polynomials?



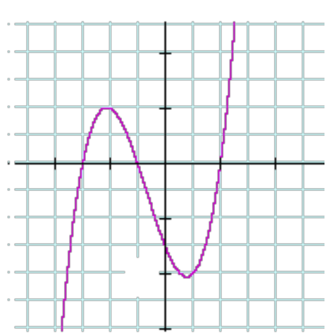
$y = ax + b$

Yes



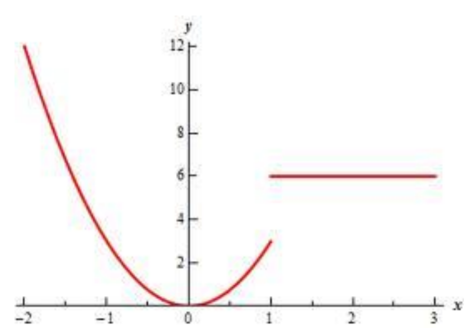
$y = ax^2 + bx + c$

Yes



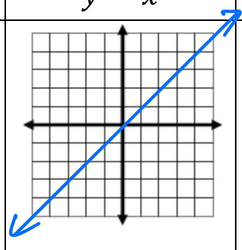
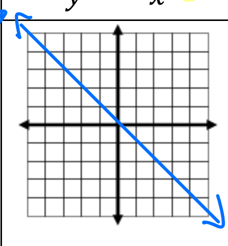
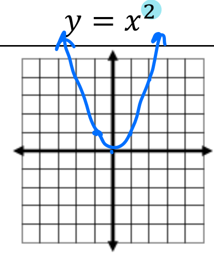
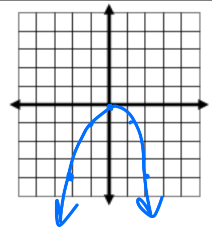
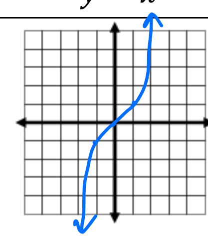
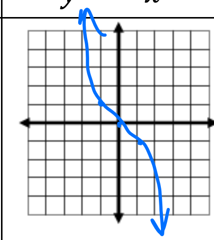
$y = ax^3 + bx^2 + cx + d$

Yes




No!

The **End Behaviour** of polynomial functions is determined by the leading coefficient $\boldsymbol{ax^n}$.

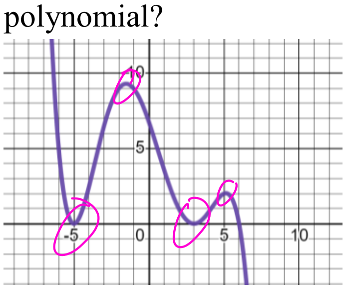
| | $y = x^1$ | $y = -x^1$ | $y = x^2$ | $y = -x^2$ | $y = x^3$ | $y = -x^3$ |
|---|---|---|---|---|---|---|
| |  |  |  |  |  |  |
| End Behaviour | Right: UP<br>Left: down<br>(Opposite) | Right: DOWN<br>Left: Up<br>(opp.) | Right: UP<br>Left: Up<br>(same) | Right: DOWN<br>Left: down<br>(same) | Right: UP<br>Left: down<br>(opp.) | Right: DOWN<br>Left: up<br>(opp.) |
| Domain | $x \in \mathbb{R}$ | $x \in \mathbb{R}$ | $x \in \mathbb{R}$ | $x \in \mathbb{R}$ | $x \in \mathbb{R}$ | $x \in \mathbb{R}$ |
| Range | $y \in \mathbb{R}$ | $y \in \mathbb{R}$ | $x \geq 0$ | $x \leq 0$ | $y \in \mathbb{R}$ | $y \in \mathbb{R}$ |
| **n** | 1 (odd) | 1 (odd) | 2 (even) | 2 (even) | 3 (odd) | 3 (odd) |
| **a** | $a > 0$ | $a < 0$ | $a > 0$ | $a < 0$ | $a > 0$ | $a < 0$ |

$\boldsymbol{n} \rightarrow$ basic shape of graph ; $\boldsymbol{a} \rightarrow$ end behaviour (always look on RIGHT side)

**Turning points** – generated by the 'middle' terms of a polynomial function.

If the degree is $\boldsymbol{n}$, the maximum number of turning points is $\boldsymbol{n-1}$ ( but could be fewer ).

**Example 2:** How many turning points are there on this graph? What is the minimum degree of the polynomial?




4 turning points

(4+1) → must be at least degree 5

**Example 3:** Determine the degree, the end behaviour and the maximum number of turning points of the following polynomial functions.

a) $f(x) = -5x^5 - 3x^4 + \sqrt{7}x^2 - 4$

$n = 5$ = degree (odd, arrows opp.)

$a = -5$ $a < 0$ ∴ ends DOWN on right, (up on left)

max 4 turning points

b) $g(x) = 2^{-3}x^8 + 5x^6 - \frac{6}{11}x^2 + 9$

$n = 8$ = degree (even, arrows SAME)

$a = 2^{-3} = \frac{1}{8}$ $a > 0$ ∴ ends UP on rt (up on left)

max 7 turning points

Assignment: p. 119 #2-3, 6a-f, 7acegi & p. 127 #1

# Polynomials Day 2: Features of Polynomials II

set y = 0 ☺

**Zero(s) of a Polynomial Function** – location(s) where the polynomial function crosses the x-axis

a) also called roots, solutions or x-intercepts

b) determined by factoring a polynomial expression

c) zeros can be real or imaginary (we only deal with real in PC 12)

set y = 0 factor! GCF? Diff. of Squares?

**Example 1:** Find all the real zeros of $f(x) = x^5 - 16x$

gcf = x

$0 = x^5 - 16x$

diff. of sq.

$0 = x(x^4 - 16)$

doesn't factor

$0 = x(x^2+4)(x^2-4)$ diff. of sq.

$0 = x(x^2+4)(x+2)(x-2)$

no real roots

$\therefore x = 0$ $x = -2$ $x = +2$

$\boxed{x = 0, \pm 2}$

A polynomial function of degree $\boldsymbol{n}$ has, at most, $n$ real zeros.

Remember, the zeros of the polynomial are really just the x-intercepts.

**Minimum and Maximum number of zeros:**

For leading term, $\boldsymbol{ax^n}$, if $\boldsymbol{n}$ is even, may cross the x-axis from 0 to $\boldsymbol{n}$ times.

E.g. $f(x) = 3x^4 - 16x^3 + 7$

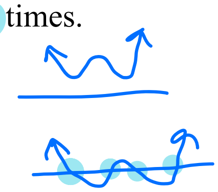
n = 4 (max)

even, min = 0

min: 0

max: 4

ex:



For leading term, $\boldsymbol{ax^n}$, if $\boldsymbol{n}$ is odd, may cross the x-axis from 1 to $\boldsymbol{n}$ times.

**(must cross at least once)**

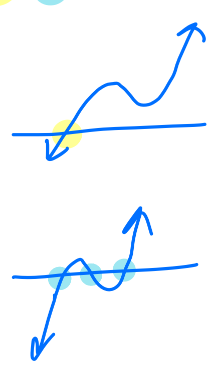
E.g. $g(x) = \sqrt{5}x^3 - 2x^2 - 3x$

n = 3 (max)

odd, min = 1

min: 1

max: 3



*must cross at least once because arrows go up/down!

YEAH

different

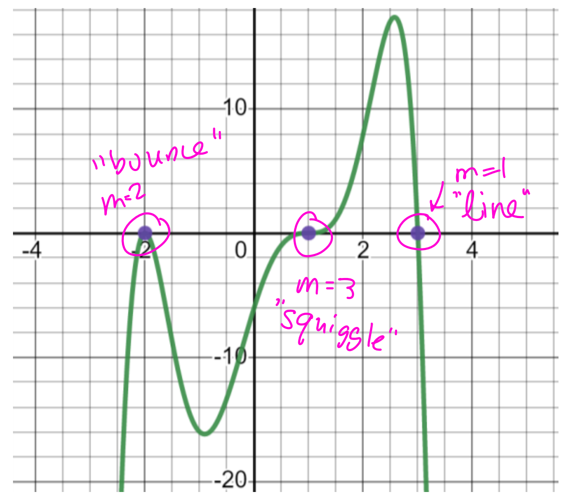
**Multiplicity** – A polynomial of degree ***n*** can have at most ***n*** distinct solutions. When a solution is repeated ***r*** times, the solution has a "multiplicity of ***r***."

**Example 2:** a) State the zeros and list the multiplicity of each zero. Then state the degree of the polynomial.

already factored :)

$$P(x) = -\frac{1}{2}(x+2)^2(x-1)^3(x-3)^1$$

← multiplicity (add up to get degree




Zeros: $x=-2$, $x=1$, $x=3$

Multiplicity: $m=2$ $m=3$ $m=1$

Degree: $2+3+1 =$ degree 6

b) The polynomial of the graph above is shown above. What do you notice about the graph at the different x-intercepts when they have different multiplicities?

different shape

↳ $m=1$ "slice"/"cut"/"line" through axis

↳ $m=2$ "bounce" off axis

↳ $m=3$ "squiggle" through axis

**Example 3:** Find all the real zeros, and the multiplicity of each zero.

a) $h(x) = -x^3 + 16x + 3x^2 - 48$ Factor by grouping

$0 = \underbrace{-x^3 + 16x}_{gcf:\,-x} + \underbrace{3x^2 - 48}_{gcf:\,3}$

$0 = -x(x^2-16) + 3(x^2-16)$

$0 = (-x+3)(\underbrace{x^2-16}_{\text{diff. of squares}})$

gcf = -1

$0 = -(x+3)^1(x+4)^1(x-4)^1$

$$\boxed{\therefore x = -3, \pm 4 \quad \text{(each has a multiplicity of 1)}}$$

b) $k(x) = -3x^4(x^2 - 4x - 5)$

$0 = -3x^4(x-5)(x+1)$

$0 = -3(x)^4(x-5)^1(x+1)^1$

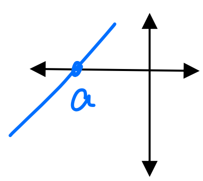
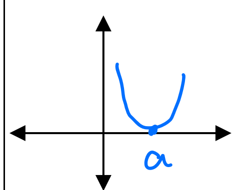
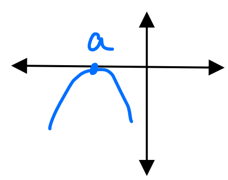
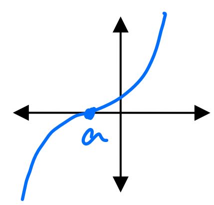
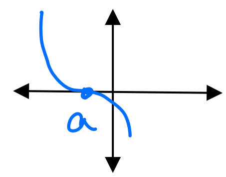
| | | |
|---|---|---|
| ∴ $x=0$ with multiplicity of 4 ($m=4$) | $x=5$ $m=1$ | $x=-1$ $m=1$ |

Assignment: p. 120 #4, #5, 9abcehijm (for 9hij use factoring by grouping)

# Polynomials Day 3: Graphing Polynomials

**Graphing a polynomial function can be summarized using the following steps:**

1. Plot the y-intercept. **(Set x = 0 and solve for y)**
2. Plot the x-intercepts. **(Find the zeros from factoring. These are the x-intercepts.)**
3. Determine the shape at each x-intercept from the multiplicities. **(See below)**

| **Multiplicity** | **n = 1**<br>"slice" | $(\boldsymbol{x-a})^{\boldsymbol{n}}$ **n is Even**<br>"bounce" | | $(\boldsymbol{x-a})^{\boldsymbol{n}}$ **n is Odd** and **>1**<br>"squiggle" | |
|---|---|---|---|---|---|
| **Factor** | $(x-a)^1$ | $(x-a)^2$ ... | $-(x-a)^2$ ... | $(x-a)^3$ ... | $-(x-a)^3$ ... |
| **Graph shape at Zero** |  |  |  |  |  |

4. If the highest power of x is positive $(\boldsymbol{a}>0)$ it ends UP on the **right**. If the highest power of x is negative $(\boldsymbol{a}<0)$ it ends DOWN on the **right**.
5. Trace backwards through all points. Make sure to "slice," "bounce" or "squiggle" at x-intercepts based on multiplicity (see step 3). (Right to left)
6. Double check: Even-degree polynomials "end" in the same direction.
   Odd-degree polynomials "end" in opposite directions.

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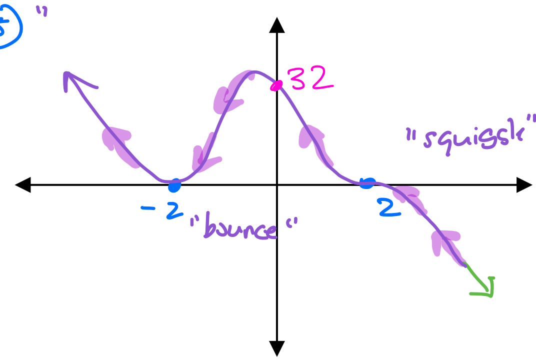
a = −1 (a is negative)

**Example 1:** Sketch: $f(x)=-(x+2)^2(x-2)^3$

⑤ "




① y-int (set x=0)

$y = -(0+2)^2(0-2)^3$

$= -(4)(-8)$

$y = +32$ (plot (0,32) at end)

② x-int (set y=0)

$0 = -(x+2)^2(x-2)^3$

$\therefore x=-2$ $x=+2$

③ (mult. of 2) (mult. of 3)

"bounce" "squiggle"

④ a = −1 ∴ a < 0 ends down on right ...

⑥ Ends go in OPPOSITE directions. Check!

degree = 2 + 3 = 5

odd degree
opp arrows

Add y-int. label at end!
Also a good check!

a Left of 0

**Example 2:** Sketch: $g(x) = (x-a)^3(x-b)(x-c)^2$ where $a < 0 < b < c$

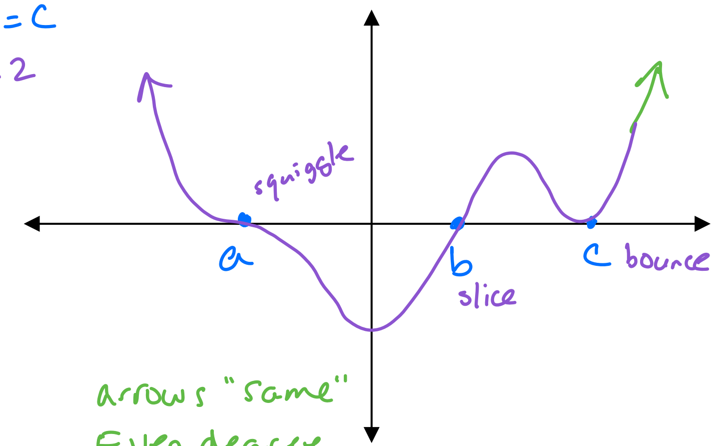
Zeros: $\therefore x = a$   $x = b$   $x = c$

m. of 3   m. 1   m. 2

degree: $3+1+2=6$

leading coeff. is positive, so UP on right

(no need to check y-int.)




arrows "same"
Even degree
So checks out!

Do Ex3 First.

**Example 4:** Determine the equation of the polynomial $g(x)$ in factored form, if $g(-2) = -12$ and the zeros are $0, 0, -1, -\frac{2}{3}$

$(x \quad)(x \quad)(x \quad)\ldots$

*Remember "a" out front!*

$g(x) = a(x-0)(x-0)(x+1)(3x+2)$

$g(x) = a\,x^2(x+1)(3x+2)$

Sub in a point to solve for "a" $g(-2) = -12$ is $(-2, -12)$ x y

$-12 = a(-2)^2(-2+1)(3(-2)+2)$

$-12 = a(4)(-1)(-4)$

$\frac{-12}{16} = \frac{16a}{16}$

$-\frac{3}{4} = a$

Equation: $\boxed{g(x) = -\frac{3}{4}x^2(x+1)(3x+2)}$

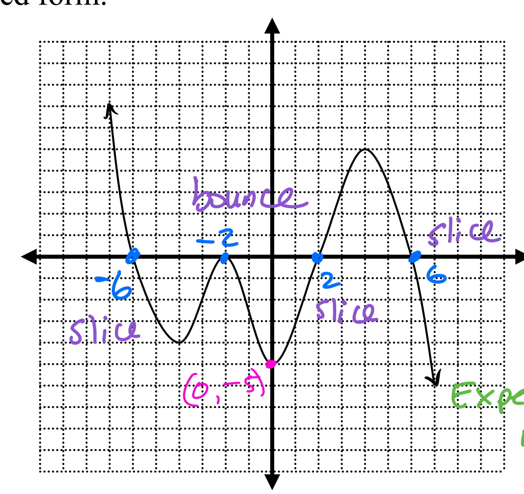
switch order :)

**Example 3:** Determine the equation of the polynomial in factored form.

$P(x) = a(x- \quad)(x \quad)(x \quad \ldots$

x-int's are $x = -6, -2, 2, 6$

"bounce" ∴ multiplicity of 2




Expect "a" negative

$P(x) = a(x+6)^1(x+2)^2(x-2)^1(x-6)^1$

Sub in a point $(0,-5)$ x y to solve for "a"

$-5 = a(0+6)(0+2)^2(0-2)(0-6)$

$-5 = a(6)(4)(-2)(-6)$

$\frac{-5}{288} = \frac{a(288)}{288}$

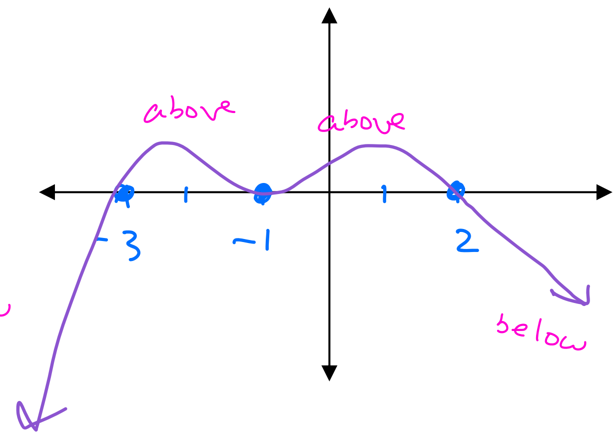
$-\frac{5}{288} = a$

$\therefore \boxed{P(x) = -\frac{5}{288}(x+6)(x+2)^2(x-2)(x-6)}$

"negative y" = negative heights = below x-axis

**Example 5:** Sketch a graph of a polynomial function of lowest degree.

| | | | | |
|---|---|---|---|---|
| Value of $f(x)$ (y) | – below | + above | + above | – below |
| Zeros of $x$ (x-int's) | | –3 | –1 | 2 |




· Plot x-int's

· Imagine where graph is "above" or "below"

· Sketch in... where does it "bounce"?

Graphing Polynomials Worksheet &

Assignment: p. 127 #2ab, 3ab, 5ab, 6, 7ab

# Polynomials Day 4: Division

Dividing a polynomial function using long division uses the same techniques as long division of numbers.

Recall: $1537 \div 12$

$$
\begin{array}{r}
128 \text{ R } 1 \\
12\overline{)1537} \\
-(12)\phantom{37} \\
33\phantom{7} \\
-(24)\phantom{7} \\
97 \\
-(96) \\
1
\end{array}
$$

← Remainder

① How many times does 12 go into 15? = 1

② $(1)\times(12) = 12$ (write below 15 and subtract)

③ Bring down next digit ↓ 3

④ Repeat (until you can no longer divide... remaining number is REMAINDER)

We can also divide polynomial expressions using these same techniques.

The result of the division of a polynomial P(x) by a binomial D(x) can be written in the forms: $\boxed{P(x) = D(x)Q(x) + R}$ and $\boxed{\frac{P(x)}{D(x)} = Q(x) + \frac{R}{D(x)}}$ where Q(x) is the quotient and R is the remainder

$1537 = (12)(128) + R$ $\qquad \frac{1537}{12} = 128 + \frac{1}{12}$

**Example 1:** Divide: $(5x^3 + 10x^2 - 13x - 9) \div (x - 2)$ (P, D)
and write final solution in both forms.

$$
\begin{array}{r}
5x^2 + 20x + 27 \text{ R } 45 \\
x-2\overline{)5x^3 + 10x^2 - 13x - 9} \\
-(5x^3 - 10x^2)\phantom{ - 13x - 9} \\
+20x^2 - 13x\phantom{ - 9} \\
-(20x^2 - 40x)\phantom{ - 9} \\
27x - 9 \\
-(27x - 54) \\
+45
\end{array}
$$

Remainder

① How many times does $x$ go into $5x^3$? $\frac{5x^3}{x} = 5x^2$

② $5x^2(x-2)$
$= 5x^3 - 10x^2$
(write below $5x^3 + 10x^2$ and subtract)
* Careful: $10x^2 - -10x^2 = 20x^2$

③ Bring down next term ↓ $-13x$

④ Repeat. Anything left is the REMAINDER.

$$\boxed{5x^3 + 10x^2 - 13x - 9 = (x-2)(5x^2 + 20x + 27) + 45}$$

$$\boxed{\frac{5x^3 + 10x^2 - 13x - 9}{(x-2)} = 5x^2 + 20x + 27 + \frac{45}{(x-2)}}$$

# Synthetic Division

An alternate (and much quicker and easier) method of dividing a polynomial by a binomial is through the use of **synthetic division**. It performs the division without writing variables and reduces the number of calculations.

P(x) ÷ D(x)

**Example 2:** Divide the following using synthetic division. $(5x^3 + 10x^2 - 13x - 9) \div (x - 2)$

Find the zero of the bracket D(x)
$x = +2$

Set up:
Write only coefficients of P(x). Put the zero of D(x) out front.

$$\begin{array}{r|rrrr} +2 & 5 & 10 & -13 & -9 \\ \hline \end{array}$$

Steps

$$\begin{array}{r|rrrr} +2 & 5 & 10 & -13 & -9 \\ & \downarrow & 10 & & \\ \hline & 5 & 20 & & \end{array}$$

① drop down the 5.
② Multiply $5 \times 2 = 10$
Write in next spot.
③ Add down the column.
$10 + 10 = 20$
④ Repeat.

(drop down the 5. Multiply by 2.
$5 \times 2 = 10$. Write 10 in next spot.

End result looks like:

$$\begin{array}{r|rrrr} +2 & 5 & 10 & -13 & -9 \\ & \downarrow & 10 & 40 & 54 \\ \hline & 5 & 20 & 27 & \boxed{45} \end{array} \leftarrow R$$

$\underbrace{5 \quad 20 \quad 27}_{Q}$

⑤ Use new #s to write coefficients of "answer" Q(x)

$Q(x) = 5x^2 + 20x + 27$

Start one power lower than $P(x) = 5x^3 \ldots$

SAME ANSWER!

$$\boxed{5x^3 + 10x^2 - 13x - 9 = (x-2)(5x^2 + 20x + 27) + 45}$$

**Example 3:** Divide the following: $(4x^3 + 7x - 5) \div (x - 3)$

use +3

* Careful! $x^2$ term is missing: $4x^3 + 0x^2 + 7x - 5$

$$\begin{array}{r|rrrr} +3 & 4 & 0 & 7 & -5 \\ & \downarrow & 12 & 36 & 129 \\ \hline & 4 & 12 & 43 & \boxed{124} \end{array} \leftarrow R$$

$\underbrace{4 \quad 12 \quad 43}$

$Q(x) = 4x^2 + 12x + 43$

$$\frac{4x^3 + 7x - 5}{(x-3)} = 4x^2 + 12x + 43 + \frac{124}{(x-3)}$$

★ NOTE: In order to do both synthetic division and long division, all descending powers of the polynomial P(x) must be present. If any power is missing from P(x), consider it as a coefficient of zero. ( e.g. $0x^2$)

What are the zeros?
Factor!

**Example 4:** Divide using synthetic division: $(2x^4 - 3x^3 - 7x^2 + 12x - 4) \div (x^2 - 4)$ and write final solution in both forms.

$(x+2)(x-2)$

zero: $-2$ zero: $+2$

start with one of them :)

$\div(x-2)$

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -7 & 12 & -4 \\ & \downarrow & 4 & 2 & -10 & 4 \\ \hline & 2 & 1 & -5 & 2 & \boxed{0} \end{array}$$

$\leftarrow R=0 \therefore (x-2)$ divides in perfectly / is a factor.

$2x^4 - 17x^3 - 7x^2 + 12x - 4 \div (x-2)$
$= 2x^3 + x^2 - 5x + 2$

Repeat process with $-2$!

$\div(x+2)$

$$\begin{array}{r|rrrr} -2 & 2 & 1 & -5 & 2 \\ & \downarrow & -4 & 6 & -2 \\ \hline & 2 & -3 & 1 & \boxed{0} \end{array}$$

$2x^2 - 3x + 1$

$2x^4 - 13x^3 - 7x^2 + 12x - 4 = (x^2-4)(2x^2-3x+1)$ OR $\dfrac{2x^4 - 13x^3 - 7x^2 + 12x - 4}{x^2 - 4} = 2x^2 - 3x + 1$

**Example 5:** Divide the following using synthetic division. $(2x^3 + 3x^2 - 5x + 2) \div (2x + 1)$

zero: $-\frac{1}{2}$

★ tricky!
★ same process until the end.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 3 & -5 & 2 \\ & \downarrow & -1 & -1 & 3 \\ \hline & 2 & 2 & -6 & \boxed{5} \end{array}$$

$(x+\frac{1}{2})\ (2x^2 + 2x - 6) + 5$

This will have a gcf!
gcf = 2

★ using $-\frac{1}{2}$ implies we $\div$ by $(x+\frac{1}{2})$

★ but we needed $(2x+1)$...
Factor a gcf out of $Q(x)$ and distribute into $D(x)$

$(x+\frac{1}{2})(2)(x^2+x-3) + 5$

$(2x+1)(x^2+x-3) + 5$ ✓

$(2x^3 + 3x^2 - 5x + 2) = (2x+1)(x^2+x-3) + 5$

Synthetic division can also be used as a quick technique to determine a missing value of a polynomial.

**Example 6:** Solve for $k$ when $x^4 + 3x^3 + kx - 1$ is divided by $x - 2$ the remainder is – 5.

$0x^2$

$$(x^4 + 3x^3 + 0x^2 + kx - 1) \div (x - 2) \qquad R = -5$$

use $+2$

$$\begin{array}{r|rrrrr} 2 & 1 & 3 & 0 & k & -1 \\ & \downarrow & 2 & 10 & 20 & 2k+40 \\ \hline & 1 & 5 & 10 & (k+20) & \boxed{2k+39} \end{array} \leftarrow R$$

$\times$

$2(k+20)$
$= 2k+40$

Remainder $= 2k+39$ also $= -5$

$\therefore \; 2k + 39 = -5$

Solve!

$$2k + 39 = -5$$
$$-39 \qquad -39$$
$$\frac{2k}{2} = \frac{-44}{2}$$
$$\boxed{k = -22}$$

Assignment p. 135 #1ac, 2acfhk, 3bcdgh, 4a-d

# Polynomials Day 5: Remainder and Factor Theorem

**The Remainder Theorem**

When a Polynomial $P(x)$ is divided by a Factor in the form $(x-a)$, the remainder is $P(a)$.

$P(a)=R$

**Example 1:**

Determine the remainder when $3x^3-5x^2+15x-9$ is divided by $x+1$

$a=-1$

$P(x)=3x^3-5x^2+15x-9$

$P(-1)=3(-1)^3-5(-1)^2+15(-1)-9$

$= -3-5-15-9$

$\boxed{P(-1)=-32}$

The remainder is $-32$.

Check with synthetic division:

$$\begin{array}{r|rrrr} -1 & 3 & -5 & 15 & -9 \\ & \downarrow & -3 & 8 & -23 \\ \hline & 3 & -8 & 23 & \boxed{-32} \end{array}$$

✓✓

**Example 2:**

Determine the remainder when $4x^3-8x^2+5x-3$ is divided by $2x+1$

$a=-\frac{1}{2}$

$P(a)=R$

$P(x)=4x^3-8x^2+5x-3$

$P(-\frac{1}{2})=4(-\frac{1}{2})^3-8(-\frac{1}{2})^2+5(-\frac{1}{2})-3$

$= -\frac{4}{8}-\frac{8}{4}-\frac{5}{2}-3$

$= -\frac{1}{2}-2-\frac{5}{2}-3$

$= -\frac{6}{2}-5$

$= -3-5$

$= -8$

$\therefore \boxed{R=-8}$

**Example 3:**

Determine $k$ if $4y^3-ky^2+3y-1 \div y-2$ has a remainder of 1.

$a=2$

$P(a)=R \quad \therefore P(2)=1$

$P(y)=4y^3-ky^2+3y-1$

$P(2)=4(2)^3-k(2)^2+3(2)-1=1$

$32-4k+6-1=1$

$37-4k=1$

$-37 \quad\quad -37$

$\frac{-4k}{-4}=\frac{-36}{-4}$

$\boxed{k=9}$

Check:

$$\begin{array}{r|rrrr} 2 & 4 & -9 & 3 & -1 \\ & \downarrow & 8 & -2 & 2 \\ \hline & 4 & -1 & 1 & \boxed{1} \end{array}$$

$R=1$ ✓✓

**The Factor Theorem**

A polynomial $P(x)$ has a factor of $x-a$ if and only if $P(a)=0$. In other words, it is a factor if the remainder is zero.

**Example 4:** Is $x-1$ a factor of $P(x)=x^3-3x^2-x+3$

Sub in $x=+1$.
if $P(1)=0$ then $(x-1)$ is a factor!

$P(x)=x^3-3x^2-x+3$

$P(1)=(1)^3-3(1)^2-(1)+3$

$=1-3-1+3$

$=0$ ✓✓ Yes! It is a factor!

**Example 5:** Is $2x+1$ a factor of $P(x)=8x^3+6x^2-9x-5$ Sub in $x=-\frac{1}{2}$

$P(x)=8x^3+6x^2-9x-5$

$P(-\frac{1}{2})=8(-\frac{1}{2})^3+6(-\frac{1}{2})^2-9(-\frac{1}{2})-5$

$=8(-\frac{1}{8})+6(\frac{1}{4})+\frac{9}{2}-5$

$=-1+\frac{3}{2}+\frac{9}{2}-5$

$=-6+\frac{12}{2}=-6+6=0$ ✓✓

Remainder $=P(a)=0$

$\therefore$ Yes! $(2x+1)$ is a factor!

Assignment p. 143 #1, 2abcd, 3abce

# Polynomials Day 6: The Rational Root Theorem
& Solving Polynomial Equations

**The Rational Root Theorem**

If $f(x) = ax^n + bx^{n-1} + cx^{n-2} + ... + jx + k$ is a polynomial function with integer coefficients, every **rational zero** of $f(x)$ is of the form $\frac{m}{n}$ where m is the possible factors of k and n is the possible factors of a.

**Example 1:** List all the possible rational zeros of the polynomial $2x^3 - 3x^2 - 14x + 15$

factors of 2: $\pm 1 \pm 2$

factors of 15: $\pm 1 \pm 3 \pm 5 \pm 15$

zeros will be of the form $\frac{m}{n} \rightarrow \frac{\pm 1 \pm 3 \pm 5 \pm 15}{\pm 1 \pm 2}$

all combinations! $\pm\frac{1}{1}$ $\pm\frac{3}{1}$ $\pm\frac{5}{1}$ $\pm\frac{15}{1}$ $\pm\frac{1}{2}$ $\pm\frac{3}{2}$ $\pm\frac{5}{2}$ $\pm\frac{15}{2}$

**Example 2:** Factor fully $x^4 - 5x^3 + 2x^2 + 20x - 24$

- First list possible rational zeros: $\pm 1 \pm 2 \pm 3 \pm 4 \pm 6 \pm 8 \pm 12 \pm 24$
- Start by testing (easy) zeros to see if any have a remainder of zero (∴ are factors).

$(x-1)$ test $x=1$: $P(1) = (1)^4 - 5(1)^3 + 2(1)^2 + 20(1) - 24$
$= 1 - 5 + 2 + 20 - 24 = -6$ ✗

$(x+1)$ test $x=-1$: $P(-1) = (-1)^4 - 5(-1)^3 + 2(-1)^2 + 20(-1) - 24$
$= 1 + 5 + 2 - 20 - 24 = -36$ ✗

$(x-2)$ test $x=2$: $P(2) = (2)^4 - 5(2)^3 + 2(2)^2 + 20(2) - 24$
$= 16 - 40 + 8 + 40 - 24 = 0$ ✓✓

Do synth. div. Get
$x^4 - 5x^3 + 2x^2 + 20x - 24 \div (x-2) = x^3 - 3x^2 - 4x + 12$

Now test $x=-2$. Use NEW ONE!
$(x+2)$ $P(-2) = (-2)^3 - 3(-2)^2 - 4(-2) + 12$
$= -8 - 12 + 8 + 12 = 0$ ✓✓

Do synth. Div again!

$(x-2)$ — 2:

| | | | | |
|---|---|---|---|---|
| 1 | -5 | 2 | 20 | -24 |
| ↓ | 2 | -6 | -8 | 24 |
| 1 | -3 | -4 | 12 | $\boxed{0}$ |

$(x+2)$ — -2:

| | | | |
|---|---|---|---|
| 1 | -3 | -4 | 12 |
| ↓ | -2 | 10 | -12 |
| 1 | -5 | 6 | $\boxed{0}$ |

$x^2 - 5x + 6$

factors more!

$(x-3)(x-2)$

Final answer: Use all factors!
$P(x) = (x-2)(x+2)(x-3)(x-2)$

$\boxed{\text{or } P(x) = (x+2)(x-2)^2(x-3)}$

**Example 3:** Factor fully $2x^3 - 5x^2 - 4x + 3$

Test $x=1$ $(\div (x-1))$

$2(1)^3 - 5(1)^2 - 4(1) + 3$

$= 2 - 5 - 4 + 3 = -4$ ✗

Test $x=-1$ $(\div (x+1))$

$2(-1)^3 - 5(-1)^2 - 4(-1) + 3$

$= -2 - 5 + 4 + 3 = 0$ ✓✓

$$\begin{array}{r|rrrr} -1 & 2 & -5 & -4 & 3 \\ & \downarrow & -2 & 7 & -3 \\ \hline & 2 & -7 & 3 & \boxed{0} \end{array}$$

$(x+1)(2x^2 - 7x + 3)$

keep factoring!

ac: $-1 \times -6 = 6$, $-1 + -6 = -7$

$2x^2 - 7x + 3$

$(x - \frac{1}{2})(x - \frac{6}{2})$

$(2x-1)(x-3)$

decomp

$2x^2 - 7x + 3$

$2x^2 - 1x - 6x + 3$

$x(2x-1) - 3(2x-1)$

$(2x-1)(x-3)$

$\boxed{= (x+1)(2x-1)(x-3)}$

**Example 4:** Solve $0 = (x-4)(x+5)(x-2)$ ✓ factored form!

Find the zeros :)

$\therefore$ $\boxed{x=4 \quad x=-5 \quad x=2}$

**Example 5:** Solve $x^3 - 9x^2 + 20x - 12 = 0$ ← need factored form. Synthetic Division!

Test $x=1$ $(\div (x-1))$

$P(1) = (1)^3 - 9(1)^2 + 20(1) - 12$

$= 1 - 9 + 20 - 12 = 0$ ✓✓

$$\begin{array}{r|rrrr} +1 & 1 & -9 & 20 & -12 \\ & \downarrow & 1 & -8 & 12 \\ \hline & 1 & -8 & 12 & \boxed{0} \end{array}$$

$(x-1)(x^2 - 8x + 12) = 0$

$(x-1)(x-2)(x-6) = 0$

$\therefore$ $\boxed{x = 1, 2, 6}$

Assignment p. 144 #4a-e, 5ace and Polynomials Supplement