

Rational Functions

Chapter Notes

Key

Assignment List

Date	Lesson	Assignment
	Features of Rational Functions I	Mickelson Page 188 #1 & Page 189 #4 (find y-intercepts as well)
	Features of Rational Functions II	Mickelson Page 181 #3
	Features of Rational Functions III	Mickelson Page 180 #2 Page 182 #4 Answer 4h) HA: $y = 2$ VA: $x = -1, 4$ hole: $(0, \frac{9}{2})$
	Graphing Rational Functions	Mickelson Page 188 #2, 5, 6
		Practice Test
		Review
		Rational Functions Test

Rational Functions Day 1:

Features of Rational Functions I

Rational Expressions

- A rational expression is a ratio of two polynomial functions.

$$\frac{g(x)}{h(x)} \text{ where } h(x) \neq 0 \quad \text{E.g. } \frac{2x-5}{3x^3+2x^2-1}$$

To simplify a single-fraction rational expression we factor the numerator and denominator fully and simplify if possible. Keep track of restrictions at each step and *state the restrictions as part of your final answer.*

Example 1: $\frac{6-2m}{m^2-9}$ ← re-order
 ← factor (difference of squares)

$$= \frac{-2m+6}{(m+3)(m-3)} \leftarrow \text{factor (gef)}$$

$$= \frac{-2(m-3)}{(m+3)(m-3)}$$

State restrictions (the "zeros" of the denom.)
 $m \neq \pm 3$
 cancel identical factor top & bottom.

$$= \frac{-2}{m+3}, m \neq \pm 3$$

Example 2: $\frac{x^2+3x-10}{2x^2+x-10}$

factor: $\frac{5}{-2} \times \frac{-2}{1} = -10$
 $\frac{1}{1} + \frac{-2}{-2} = -1$

factor: $\frac{5}{1} \times \frac{-2}{-1} = -10$
 $\frac{1}{1} + \frac{-2}{-2} = 1$

$$= \frac{(x+5)(x-2)}{(2x+5)(x-2)}$$

Restrictions:
 $x \neq -\frac{5}{2}, 2$
 cancel

$$= \frac{x+5}{2x+5}, x \neq -\frac{5}{2}, 2$$

*note: nothing else cancels!

Restrictions

- Restrictions** are any values of x that would make the denominator equal to zero. *Restrictions are part of the final answer.* They also help us describe the **Domain** of a **Rational Function**.
- Domain** describes all the possible x -values of a function.

Rational Functions

- A **rational function** is an equation with "x" and "y" (or $f(x)$) that has a rational expression in it. Until we have better techniques, we can graph a rational function using a table of values.

E.g. $f(x) = \frac{2x-5}{3x^3+2x^2-1}$

x- and y- intercepts

- x-intercepts** are where the graph crosses the x-axis (where height = 0, or $y = 0$).
- y-intercepts** are where the graph crosses the y-axis (where you are "zero over" or $x = 0$).

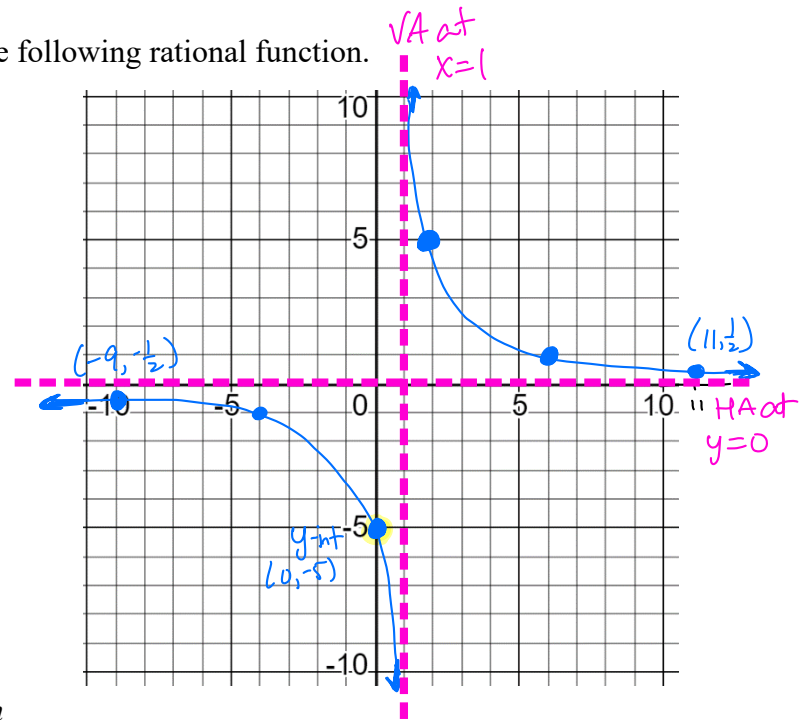
Graphs of Rational Functions

Example 3: a) Create a table of values to graph the following rational function.

$$f(x) = \frac{5}{x-1} \quad \text{sub in } x = -9 :$$

x	y
-9	$-\frac{1}{2}$
-4	-1
0	-5
1	(undefined)
2	5
6	1
11	$\frac{1}{2}$

$y = \frac{5}{(-9)-1} = \frac{5}{-10} = -\frac{1}{2}$ etc.
 $y = \frac{5}{(-4)-1} = \frac{5}{-5} = -1$
 $y = \frac{5}{(0)-1} = \frac{5}{-1} = -5$
 ← restriction: $x \neq 1$
 $y = \frac{5}{(2)-1} = \frac{5}{1} = 5$
 $y = \frac{5}{(6)-1} = \frac{5}{5} = 1$
 $y = \frac{5}{(11)-1} = \frac{5}{10} = \frac{1}{2}$



b) label any x- and y-intercepts on the graph

no x-int's. y-int. at (0, -5)

c) List any restrictions: $x \neq 1$ d) what is the Domain? $D: \{x \mid x \neq 1, x \in \mathbb{R}\}$
 (x is all real numbers except for 1.)

e) What happens as the x-values get really close to 1?

test: $x = 0.9$ $y = \frac{5}{(0.9)-1} = -50$ Big neg #'s! ↓ test: $x = 1.1$ $y = \frac{5}{(1.1)-1} = 50$ Big #'s! ↑

f) What happens as we put in really large, positive values for x?

test: $x = 100$ $y = \frac{5}{(100)-1} = 0.05$ small! Close to zero (but positive)

g) What happens as we put in really large, negative values for x?

test: $x = -100$ $y = \frac{5}{(-100)-1} = -0.049$ small! Close to zero (but negative)

Asymptotes

- An **asymptote** is a line which the graph approaches but never reaches. We use a dotted line to show them on a graph. (Draw in asymptotes on the graph above)

- Vertical Asymptotes (VA)** have an equation of the form " $x = _$ "

(Label the vertical asymptote on the graph above)

$$x = 1$$

- Horizontal Asymptotes (HA)** have an equation of the form " $y = _$ "

(Label the horizontal asymptote on the graph above)

$$y = 0$$

Determining y-intercepts Algebraically

1. Set $x = 0$
2. Find y

Hint: remember that $f(x) = y$

Example 4: Find the **y-intercepts** of the following Rational Functions:

a) $f(x) = \frac{x^2+7x+10}{x^2-4}$ (set $x=0$)

$$y = \frac{[(0)^2 + 7(0) + 10]}{[(0)^2 - 4]}$$

$$y = \frac{10}{-4}$$

$$y = -5/2$$

$$(0, -5/2)$$

b) $g(x) = 1 - \frac{8}{x^2-1}$ (set $x=0$)

$$y = 1 - \frac{8}{(0)^2-1}$$

$$y = 1 - \frac{8}{-1}$$

$$y = 1 + 8$$

$$y = 9$$

$$(0, 9)$$

Determining x-intercepts (also called "the zeros") Algebraically

1. Set $y = 0$ Hint: remember that $f(x) = y$
2. Solve for x

To solve a rational equation:

1. Factor everything. State restrictions. Reduce if possible.
2. "Clear denominators" (multiply all terms by any factor in a denominator).
3. Solve the resulting equation.
 \rightarrow terms are separated by add/subtract

Example 5: Find the **x-intercepts** of the following Rational Functions:

a) $f(x) = \frac{x^2+7x+10}{x^2-4}$ (set $y=0$)
factor!

$$0 = \frac{(x+5)(x+2)}{(x+2)(x-2)}$$

Restrictions:
 $x \neq \pm 2$
Then cancel

$$(x-2) \cdot 0 = \frac{(x+5)}{(x-2)} \cdot (x-2)$$

Clear fractions

$$0 = x+5$$

solve

$$-5 = x$$

$$x\text{-int: } (-5, 0)$$

b) $g(x) = 1 - \frac{8}{x^2-1}$ (set $y=0$)
factor

$$0 = 1 - \frac{8}{(x+1)(x-1)}$$

Restrictions:
 $x \neq \pm 1$
Nothing cancels.

$$0 \cdot (x+1)(x-1) = \frac{(x+1)(x-1)}{(x+1)(x-1)} - \frac{8}{(x+1)(x-1)}$$

Clear fractions

$$0 = (x+1)(x-1) - 8$$

Expand & collect

$$0 = x^2 - 1 - 8$$

$$0 = x^2 - 9$$

$$0 = (x+3)(x-3)$$

factor to solve quadratic

$$\therefore x = \pm 3$$

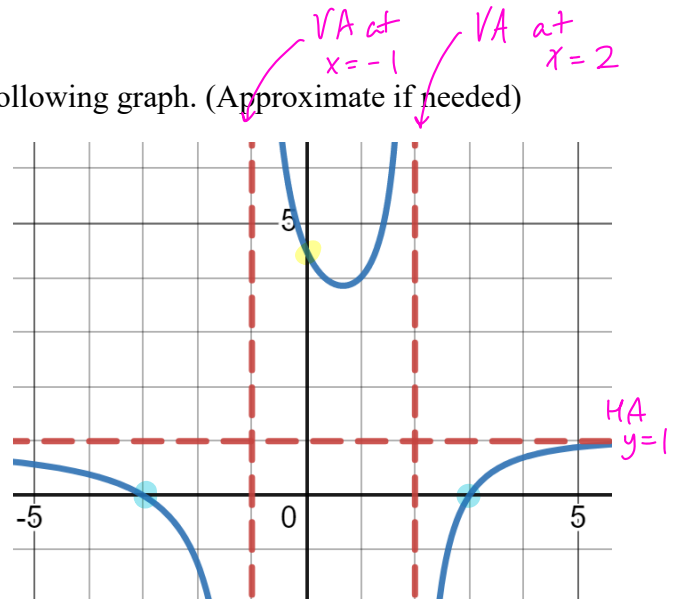
$$x\text{-ints: } (-3, 0) \text{ and } (3, 0)$$

Assignment: Page 188 #1, #4 (#4 asks for "zeros" which are x-intercepts. Find y-intercepts as well)

Rational Functions Day 2:

Features of Rational Functions II

Warm-up: Label any intercepts and asymptotes on the following graph. (Approximate if needed)



x-intercept(s): $(-3, 0), (3, 0)$

y-intercept: $(0, 4.5)$ *approx. ?*

HA: $y = 1$

VA: $x = -1, x = 2$

Domain: $\{x \mid x \neq -1, 2, x \in \mathbb{R}\}$

Example 1: The equation of the graph above is $f(x) = \frac{x^2 - 9}{x^2 - x - 2}$.

Can confirm y-int: $y = \frac{0^2 - 9}{0^2 - 0 - 2} = \frac{-9}{-2} = 4.5$

a) Factor and simplify. State restrictions.

$$f(x) = \frac{(x+3)(x-3)}{(x+1)(x-2)}, \quad x \neq -1, 2$$

b) What do you notice about the restrictions and the asymptotes? *The same!*

Except the restrictions and Domain are $x \neq -1, 2$ and the asymptotes are at $x = -1, 2$

Determining the Vertical Asymptotes Algebraically

- The vertical asymptotes are created by restrictions from the denominator (after simplifying)
- For rational function, $f(x) = \frac{g(x)}{h(x)}$, if c is a zero of $h(x)$, then the vertical line $x = c$ is a vertical asymptote of $f(x)$.

Example 2: Determine the vertical asymptotes of $f(x) = \frac{2x-5}{x^3-9x^2}$

$$f(x) = \frac{2x-5}{(x)^2(x-9)}, \quad x \neq 0, 9$$

factor restrictions cancel? No. VA at restrictions (that don't cancel)

$$\therefore \text{VA at } x = 0, 9$$

Determining Horizontal Asymptotes Algebraically

- Horizontal asymptotes can be determined by examining the end behaviour of a rational function as x approaches positive and negative infinity (gets really big or really small).

- To examine the end behaviour, divide each term in the rational expression by the highest power of x and allow $|x| \rightarrow \infty$. This allows all but the leading term(s) to become zero and disappear.

Caution: Constants beyond the rational function get added to the value of the horizontal asymptote.

Add shortcuts after:

"bottom wins" $\rightarrow 0$

"tied" \rightarrow coefficients

"top wins" \rightarrow no HA

Example 3: Determine the horizontal asymptotes of

i) $f(x) = \frac{2x-5}{x^3-9x^2} + 3$

$$y = \frac{\frac{2x}{x^3} - \frac{5}{x^3}}{\frac{x^3}{x^3} - \frac{9x^2}{x^3}} + 3$$

$$y = \frac{\frac{2}{x^2} - \frac{5}{x^3}}{1 - \frac{9}{x}} + 3$$

$$y = \frac{0-0}{1-0} + 3 = \frac{0}{1} + 3$$

Short-cut!

For Horizontal Asymptotes:

For $f(x) = \frac{g(x)}{h(x)} = \frac{a_n x^n + \dots + a_1 x + a_0}{b_d x^d + \dots + b_1 x + b_0}$ where $a_n \neq 0, b_d \neq 0$

"bottom wins" $\rightarrow 0$

1. If $n < d$, the line $y = 0$ (the x -axis) is a horizontal asymptote. (highest power is in denom.)

"tied" \rightarrow coefficients

2. If $n = d$, the line $y = \frac{a}{b}$ is a horizontal asymptote. (highest power is found in num. & den.)

"top wins" \rightarrow no HA

3. If $n > d$, there is no horizontal asymptote. (highest power is in numerator)

ii) $g(x) = \frac{6x^2+2x-1}{4x^2-9x+7}$

$$y = \frac{\frac{6x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}{\frac{4x^2}{x^2} - \frac{9x}{x^2} + \frac{7}{x^2}}$$

$$y = \frac{6 + \frac{2}{x} - \frac{1}{x^2}}{4 - \frac{9}{x} + \frac{7}{x}}$$

$$y = \frac{6}{4} = \frac{3}{2}$$

HA @ $y = \frac{3}{2}$

as x gets really big, $\frac{1}{x} \rightarrow 0$

iii) $h(x) = \frac{3x^2-1}{9x+4} - 3$

$$y = \frac{\frac{3x^2}{x^2} - \frac{1}{x^2}}{\frac{9x}{x^2} + \frac{4}{x^2}} - 3$$

$$y = \frac{3 - \frac{1}{x^2}}{\frac{9}{x} + \frac{4}{x^2}} - 3$$

$$y = \frac{\frac{3}{0}}{0} - 3$$

! undefined!

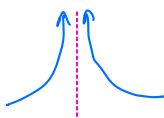
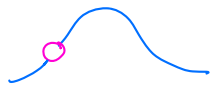
\therefore no H.A.

(note: "undefined" - 3 is still undefined!)

Rational Functions Day 3:

Features of Rational Functions III

Discontinuity

- Rational functions are **undefined** when there is a value of x that causes **division by zero**. This causes a discontinuity or “break” of some kind in the graph.
- Vertical Asymptotes** are a type of discontinuity we have already seen. 
- Holes** are another type of discontinuity. 

Holes in Rational Functions

- A **hole** in a rational function is created when the factored form of a rational function **simplifies** by **cancelling** common factors. But because the **restricted value still exists**, a **hole** in the function is created at that x -value.
- To find the coordinate of the hole, sub the restricted value (that cancelled away) into the **simplified** equation to find y .

Example 1: Consider $f(x) = \frac{x+2}{x^2+x-2}$. The graph is shown below.

- a) Simplify the function and state restrictions.

$$f(x) = \frac{(x+2)}{(x+2)(x-1)}, \quad x \neq -2, 1$$

$$f(x) = \frac{1}{x-1}, \quad x \neq -2, 1$$

- b) Explain why there is an asymptote at $x = 1$ and a hole at $x = -2$.

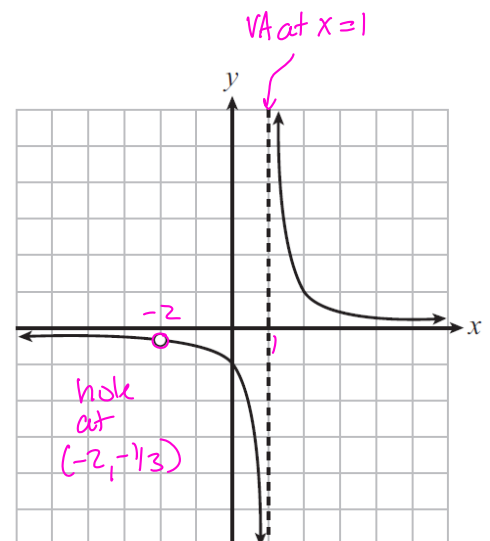
- The factor $(x-1)$ is still in the simpler function, and creates a V.A.
- The factor $(x+2)$ cancelled, but $x \neq -2$ is still a restriction.

- c) Sub $x = -2$ into the **simplified** version of $f(x)$ to find the coordinate of the hole:

$$f(-2) = \frac{1}{(-2)-1} = \frac{1}{-3} = -\frac{1}{3}$$

hole at $(-2, -\frac{1}{3})$

Looks about right on the graph!



Features of Rational Functions

Now we are able to find all the key features of a rational function. Next day we can graph them!

Example 2: Determine the vertical & horizontal asymptotes, x & y intercepts, and any holes in the function

$$f(x) = \frac{x^2 - 5x}{x^3 - 25x} - 2$$

HA

denom. has highest power

$$y = \frac{x^2 - 5x}{x^3 - 25x} - 2$$

$$y = 0 - 2$$

HA at $y = -2$

Simplify:

$$y = \frac{x^2 - 5x}{x^3 - 25x} - 2 \text{ factor}$$

$$y = \frac{x(x-5)}{x(x^2-25)} - 2$$

$$y = \frac{x(x-5)}{x(x+5)(x-5)} - 2, \quad x \neq 0, -5, 5$$

\downarrow hole \downarrow VA \downarrow hole

Simplified Equation (use for the other work)

$$y = \frac{1}{x+5} - 2, \quad x \neq 0, -5, 5$$

VA

VA at $x = -5$

Holes

Sub in $x=0$ and $x=5$ into simplified eqn

$$y = \frac{1}{(0)+5} - 2$$

$$y = \frac{1}{5} - 2$$

$$y = \frac{1}{5} - \frac{10}{5}$$

$$y = -\frac{9}{5}$$

hole at $(0, -9/5)$

$$y = \frac{1}{(5)+5} - 2$$

$$y = \frac{1}{10} - 2$$

$$y = \frac{1}{10} - \frac{20}{10}$$

$$y = -\frac{19}{10}$$

hole at $(5, -19/10)$

y-int

Set $x=0$

! but $x \neq 0$

\therefore no y-int

x-int

Set $y=0$ (use simplified!)

$$0 = \frac{1}{x+5} - 2$$

clear fractions

$$0(x+5) = \frac{1(x+5)}{(x+5)} - 2(x+5)$$

$$0 = 1 - 2(x+5)$$

$$0 = 1 - 2x - 10$$

Expand

Collect

$$0 = -2x - 9$$

$$\frac{9}{-2} = \frac{-2x}{-2}$$

$$-\frac{9}{2} = x$$

x-int: $(-9/2, 0)$

Solve

Example 3: Determine the vertical & horizontal asymptotes, x & y intercepts, and any holes in the function

$$f(x) = \frac{9x - x^3}{2x^3 - 3x^2 - 9x}$$

HA

powers "tied" top/bottom:
∴ use coefficients!

$$\frac{-1x^3}{2x^3}$$

∴ HA at $y = -\frac{1}{2}$

Simplify:

$$y = \frac{-x(x^2 - 9)}{x(2x^2 - 3x - 9)}$$

$$y = \frac{-x(x+3)(x-3)}{x(2x+3)(x-3)}, \quad \begin{matrix} x \neq 0, -\frac{3}{2}, 3 \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{hole} \quad \text{VA} \quad \text{hole} \end{matrix}$$

Simplified Eq'n

$$y = \frac{-(x+3)}{(2x+3)}, \quad x \neq 0, -\frac{3}{2}, 3$$

VA VA at $x = -\frac{3}{2}$

y-int (set $x=0$, but $x \neq 0$!)
∴ no y-int

Holes at $x=0$ and $x=3$

x-int (set $y=0$)

$$y = \frac{-(0+3)}{(2(0)+3)}$$

$$y = \frac{-(3+3)}{2(3)+3}$$

$$0 = \frac{-(x+3)}{(2x+3)} \quad \text{clear fractions}$$

$$y = \frac{-(3)}{3}$$

$$y = \frac{-6}{(6+3)}$$

$$0 \stackrel{(2x+3)}{=} \frac{-(x+3)(2x+3)}{(2x+3)}$$

can expand, or use the "zero of the bracket!"

$$y = -1$$

$$y = \frac{-6}{9}$$

$$0 = -(x+3)$$

$$x = -3$$

Hole at $(0, -1)$

$$y = -\frac{2}{3}$$

Hole at $(3, -\frac{2}{3})$

x-int: $(-3, 0)$

Rational Functions Day 4:

Graphing Rational Functions

Now that we know how find all the key features of a rational function, we can use them to graph!

Key Ideas:

- Restrictions
- Asymptotes
- Intercepts
- Holes

★ Sometimes, when we simplify a function, it turns into something we know how to graph already. Just watch out for restrictions!

Example 1: Sketch an accurate graph of $f(x) = \frac{x^2-9}{x+3}$.

HA

top has highest power
∴ no HA

Simplify:

factor

$$y = \frac{(x+3)(x-3)}{(x+3)}$$

$x \neq -3$

↑
hole

no VA

VA

no VA

$$y = x - 3, \quad x \neq -3$$



• this is a line!

$$y = mx + b$$

$$y = x - 3$$

↑ ↗ b = -3
m = 1

• except a line with a hole in it!

hole

$$x = -3$$

$$y = x - 3$$

$$y = (-3) - 3$$

$$y = -6$$

hole at $(-3, -6)$

(Plot)

plot points on line:

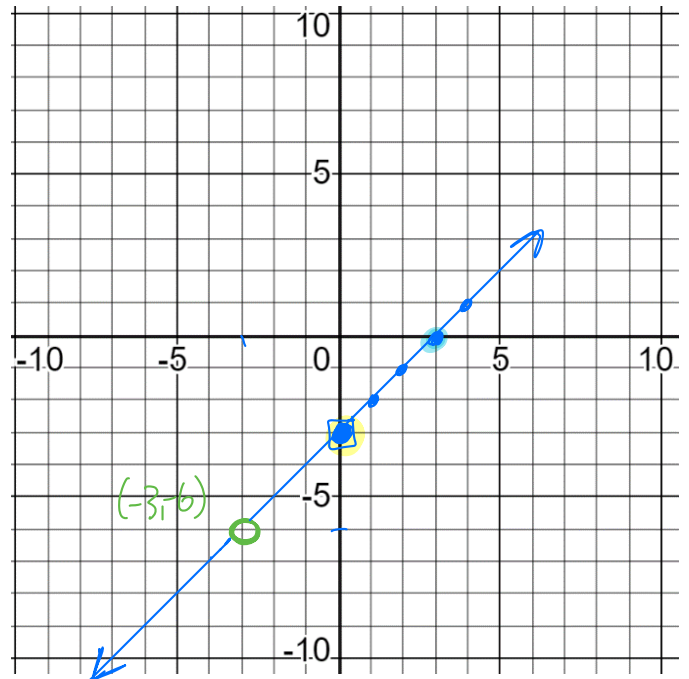
y-int at -3

up rt!

x-int: $(3, 0)$

y-int: $(0, -3)$

(from graph)



Steps to Graphing a Rational Function:

1. Find and plot any **Horizontal Asymptotes** (before you simplify)
2. **Simplify** the function* (factor, restrictions, cancel – note Holes/VA)

*Note: You may be able to sketch the graph of the function in its simplified form if you know it (e.g. a line). If not....

Use the **simplified** function to find all of the following:

3. Find and plot any **Vertical Asymptotes**
4. Find and plot **y-intercept** (set $x = 0$ and find y) and **x-intercepts** (set $y = 0$ and solve for x)
5. Find the y-coordinates of any holes (sub x -value of hole into **simplified** function) and plot **holes**
6. You may need to use a table of values to include
 - a. $x = 0.1$ or 0.01 above and below each vertical asymptote (to see how it approaches)
 - b. $x = \pm 1000$ ($\pm\infty$) (to see end behaviour)
 - c. any other helpful points
7. Draw a smooth curve through the points, and “approach” the asymptotes

Example 2: Sketch an accurate graph of $f(x) = \frac{x-2}{x+1}$.

(No factoring)

Simplified Equation $y = \frac{(x-2)}{(x+1)}$, $x \neq -1$ (VA)

HA Powers “tied” \therefore use coefficients
HA at $y = 1$

VA VA at $x = -1$

Holes no restrictions that cancelled \therefore
no holes

y-int set $x = 0$
 $y = \frac{(0-2)}{(0+1)}$
 $y = \frac{-2}{1}$
y-int: $(0, -2)$

x-int set $y = 0$
 $(x+1) \cdot 0 = (x-2)(x+1)$
 $0 = (x-2)$
 $x = 2$
x-int: $(2, 0)$

Other Points Need to know what's happening “left of VA at $x = -1$ ”
try $x = 1$ plot $(1, -\frac{1}{2})$
 $y = \frac{(1-2)}{(1)+1} = \frac{-1}{2}$
try $x = -2$ plot $(-2, 4)$
 $y = \frac{(-2)-2}{(-2)+1} = \frac{-4}{-1} = 4$
try $x = -4$ plot $(-4, 2)$
 $y = \frac{(-4)-2}{(-4)+1} = \frac{-6}{-3} = 2$

Connect dots, “follow” asymptotes

Example 3: Sketch an accurate graph of $f(x) = \frac{x^2+5x+6}{x^2-9}$.

highest powers "tied". Use coeff.

HA

$y = 1$
HA at $y = 1$

VA

VA at $x = 3$

Simplify: $y = \frac{(x+2)(x+3)}{(x+3)(x-3)}$, $x \neq -3, 3$
hole VA

Simplified Eq'n: $y = \frac{(x+2)}{(x-3)}$, $x \neq -3, 3$

hole

Sub $x = -3$ into simpler eq'n

$y = \frac{(-3+2)}{(-3)-3}$

$y = \frac{-1}{-6}$

$y = \frac{1}{6}$ ∴ hole at $(-3, 1/6)$

y-int: set $x = 0$

$y = \frac{(0+2)}{(0-3)}$

$y = \frac{2}{-3}$

$y = -2/3$

y-int at $(0, -2/3)$

x-int: set $y = 0$

$(x-3) \cdot 0 = \frac{(x+2)(x-3)}{(x-3)}$

x-int at $(-2, 0)$

$0 = (x+2)$

$x = -2$

Other Points

What's happening right of VA?

try $x = 4$

$y = \frac{(4+2)}{(4-3)} = \frac{6}{1} = 6$ Plot $(4, 6)$

try $x = 5$

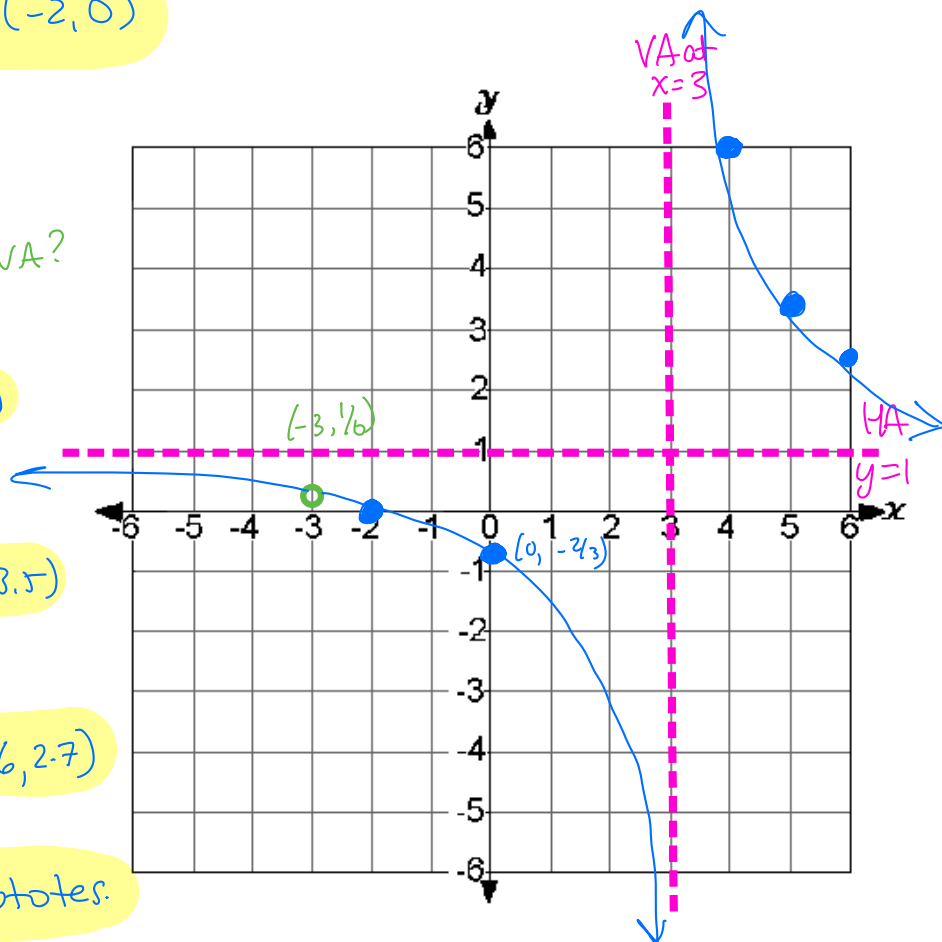
$y = \frac{(5+2)}{(5-3)} = \frac{7}{2} = 3.5$ Plot $(5, 3.5)$

try $x = 6$

$y = \frac{(6+2)}{(6-3)} = \frac{8}{3} = 2.\bar{6}$ Plot $(6, 2.\bar{6})$

Draw curves; follow asymptotes.

* draw curve as though it is going "through" the hole, but just leave the hole out!



Example 4: Sketch an accurate graph of $f(x) = \frac{2x}{x^2-4}$.

Simplify: $y = \frac{2x}{(x+2)(x-2)}$, $x \neq -2, 2$
↑ VA ↑ VA

y-int: set $x=0$
 $y = \frac{2(0)}{(0+2)(0-2)}$
 $y = 0$

y-int: (0,0)

x-int: Set $y=0$
 $(x+2)(x-2) \cdot 0 = \frac{2x}{(x+2)(x-2)}$
 $0 = \frac{2x}{2}$
 $0 = x$

x-int (0,0)

Extra Points Need to know what's happening in each "section"

x	y	$y = \frac{2x}{(x+2)(x-2)}$
-3	-1.2	$\frac{2(-3)}{(-1+2)(-3-2)} = \frac{-6}{(1)(-5)} = \frac{-6}{-5} = 1.2$
-1	0.7	$\frac{2(-1)}{(-1+2)(-1-2)} = \frac{-2}{(1)(-3)} = \frac{-2}{-3} = 0.67$
1	-0.7	$\frac{2(1)}{(1+2)(1-2)} = \frac{2}{(3)(-1)} = \frac{2}{-3} = -0.67$
3	1.2	$\frac{2(3)}{(3+2)(3-2)} = \frac{6}{(5)(1)} = \frac{6}{5} = 1.2$
-1.5	1.7	} Typed in calculator
1.5	-1.7	

highest power in denom. ∴

HA HA at $y=0$

*Note: graphs can cross HA's in the middle. They only apply at the "ends".

VA VA's at $x=-2, x=2$

holes no holes

