PC12 Name____________________________

Sequences and Series Chapter Notes

Key

Assignment List

Date	Lesson	Assignment
	Geometric Sequences	Mickelson Page 22 #1a-h, 2a-e, 3ab, 4acefghi, 13-16 (extension: complete all of #1-4)
	Geometric Series	Mickelson Page 28 #1 (omit h), 7-8, 12-14, 16, 19
	Infinite Geometric Series	Mickelson Page 35 #1, 2, 4, 12, 15, 16, 20
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Sequences and Series Day 1: Geometric Sequences

What is a sequence?

A **sequence** is simply an ordered list of numbers (called terms) that follow a pattern so that the next term can be determined.

The **first term** in the sequence is labeled a (or t_1)

The **number of terms** in the sequence is n

t_n

Any term of the sequence is t_n (read *t* sub *n*), dependent on the value of ***n***. For example, the **third term** is t_3, the **eighth term** is t_8, etc.

A finite **sequence** has a finite number of terms whereas an infinite **sequence** has an infinite number of terms

A **geometric sequence** is a sequence in which the ratio of consecutive terms is constant.

Suppose you have the sequence 4, 12, 36, 108, ...

$\times 3$ $\times 3$ $\times 3$

t_1 t_2 t_3 t_4 t_5

$\times 3$

a) What is *a*?

$a = 4$

b) What do you multiply by to get the next term (this is the ***r*** value)?

$r = 3$

c) Is the sequence geometric (see the definition above)? In other words, is the ***r*** value consistent throughout the sequence?

Yes! r is always 3.

d) What is t_5? Explain how you got t_5. Write a general formula for this.

$t_5 = 108 \times 3 = 324$ / $t_5 = t_4 \times r$

e) Show how to get t_5 using only *a* and *r*.

$t_5 = ar^4$

f) Show how to get t_8 using only a and r.

$$t_8 = ar^7$$

g) What do you notice about the exponent on r compared to n?

The exponent is one less than the term number.

h) Write a general formula for t_n for any geometric sequence:

$$t_n = ar^{n-1}$$

The general term of a geometric sequence where n is a positive integer is: $$t_n = ar^{n-1}$$ For any integer $n \geq 1$ where a is the first term, n is the number of terms, r is the common ratio, and t_n is a general term

For a geometric sequence, the common ratio (***r***), can be found by taking any term (except the first) and dividing that term by the preceding term. So $\boxed{r = \frac{t_n}{t_{n-1}}}$ ← any term ← the term before

Example 1: Are the following sequences geometric (i.e. Is the ***r*** value consistent)?

a) 2, 4, 6, 8 ∴ not geometric

$\overset{?}{r} = \frac{4}{2} = 2$ $\overset{?}{r} = \frac{6}{4} = 1.5$

not the same r

b) 4, 10, 25, 62.5 ✓ ∴ geometric

$r = \frac{10}{4} = 2.5$ / $r = \frac{25}{10} = 2.5$ / $r = \frac{62.5}{25} = 2.5$

(same r)

Example 2: In a geometric sequence, the second term is 28 and the fifth term is 1792. Determine the values of a and r, and list the first three terms of the sequence.

$\underline{\quad}_{t_1}$ $\underline{28}_{t_2}$ $\underline{\quad}_{t_3}$ $\underline{\quad}_{t_4}$ $\underline{1792}_{t_5}$

$\times r$ $\times r$ $\times r$

⇒ $28 \times r^3 = 1792$

$\frac{28 \times r^3}{28} = \frac{1792}{28}$

$r^3 = 64$

$\sqrt[3]{r^3} = \sqrt[3]{64}$

$\boxed{r = 4}$

solve for r!

if $r = 4$,

$t_1 = t_2 \div 4$

$= 28 \div 4$

$t_1 = 7$

$\boxed{a = 7}$

$\boxed{7, 28, 112}$

Applications

If a question involves percent **growth**, ***r*** must be greater than 1.

E.g. If there is 30% growth each year, what is the ***r*** value for the problem? $\boxed{r = 1.30}$

30% + 100% = 130%

If a question involves a percent reduction, ***r*** must be less than 1 and must represent the percent remaining (not the percent lost).

If you reduce the size of your savings by 25% per year, what is ***r***? $\boxed{r = 0.75}$

100% − 25% = 75%

If you increase an image by 10%, what is ***r***?

100% + 10% = 110% $\boxed{r = 1.1}$

Example 3: Bacteria reproduce by splitting into two. Suppose there were three bacteria originally present in a sample. How many bacteria will there be after 8 generations?

$r = 2$

$a = 3$

$n = 8$

$3, 6, 12, \ldots \_\_\_$

t_1 t_2 t_3 t_8

1st gen 2nd gen 8th gen

$a = 3$
$r = 2$
$n = 8$

There will be 384 bacteria.

$t_n = ar^{n-1}$

$t_8 = 3(2)^{8-1}$

$= 3(2)^7$

$= 3(128)$

$\boxed{t_8 = 384}$

Example 4: Suppose a photocopier can reduce a picture to 60% of its original size. If the picture is originally 42cm long, what length will it be after five successive reductions?

$\boxed{r = 0.60}$ (need % remaining)

$\boxed{a = 42}$

$\boxed{n = 6}$

$\underset{t_1}{42}, \_\_\__{t_2}, \_\_\__{t_3}, \_\_\__{t_4}, \_\_\__{t_5}, \_\_\__{t_6}$

1 reduction 2 3 4 5

The length will be 3.27cm.

$t_n = ar^{n-1}$

$= 42(0.6)^{6-1}$

$= 42(0.6)^5$ 5 reductions :)

$= 3.26592$

$\boxed{t_6 = 3.27\text{cm}}$

Assignment p. 22 #1a-h, 2a-e, 3ab, 4acefghi, 13-16

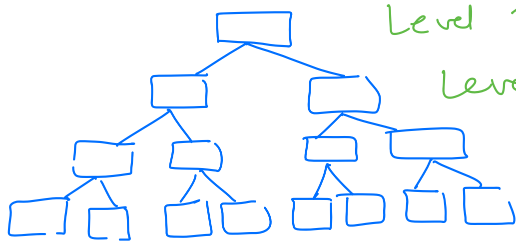
(extension: complete all of #1-4)

$S_4 = 1 + 2 + 4 + 8 \ \ldots$

Sequences and Series Day 2: Geometric Series

Geometric Series: The Sum of a geometric sequence.

One example of a geometric series is a phone tree. Suppose the school band had to share information that they didn't want public, so they used a phone tree. Draw the first four levels of the tree below:



Level 1 $t_1 = 1$

Level 2 $t_2 = 2$

level 3 $t_3 = 4$

level 4 $t_4 = 8$

What pattern has developed? $1 + 2 + 4 + 8 + \ldots$

What is the common ratio of the sequence?

$r = 2$

The sum of a geometric series can be determined using the formula

$$S_n = \frac{a(r^n - 1)}{(r-1)}, r \neq 1 \qquad \text{OR} \qquad S_n = \frac{(rt_n - a)}{(r-1)}, r \neq 1$$

last term

where a is the first term, n is the number of terms, r is the common ratio, t_n is the last term, and S_n is the sum of the first n terms

in total → sum

Use the formula(s) to find the ~~amount~~ number of people reached in the school band after 7 layers of the phone tree:

$1 + 2 + 4 + 8 + \ldots$

$a = 1$

$r = 2$

$n = 7$

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

$$S_n = \frac{1(2^7 - 1)}{(2-1)}$$

$$= \frac{1(128 - 1)}{1}$$

$\boxed{S_7 = 127}$

127 people were contacted after 7 layers.

Example 1: Determine the sum of the first 8 terms of the following geometric series.

n

a) 5 + 15 + 45 + …

b) $a = 64$, $r = \frac{1}{4}$

a) $5, 15, 45, \ldots$ $\times r$

$a = 5$
$r = 3$
$n = 8$

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

$$= \frac{5(3^8 - 1)}{(3-1)}$$

$$= \frac{5(6561-1)}{2}$$

$$= \frac{5(6560)}{2}$$

$$\boxed{S_8 = 16\,400}$$

b) $64 + 16 + 4 + \ldots$

$a = 64$
$r = \frac{1}{4} = 0.25$
$n = 8$

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

$$= \frac{64(0.25^8 - 1)}{(0.25 - 1)}$$

$$\boxed{S_8 = 85.332}$$

Example 2: Determine the sum of the following geometric series.

a) $\frac{1}{64} + \frac{1}{16} + \frac{1}{4} + \cdots + 1024$ (a, t_n)

b) $-2 + 4 - 8 + \cdots - 8192$

a) $r = \frac{t_n \text{ (any term)}}{t_{n-1} \text{ (term before)}}$

$$= \frac{1/4}{1/16} = \frac{1}{4} \times \frac{16}{1} = 4$$

$a = \frac{1}{64}$
$r = 4$
$t_n = 1024$

$$S_n = \frac{(r t_n - a)}{(r-1)}$$

$$= \frac{\left[(4)(1024) - \frac{1}{64}\right]}{(4-1)}$$

$$\boxed{S_n = 1365.328}$$

b) $-2, 4, -8, \ldots$ ($\times -2$, $\times -2$)

$(-2) + 4 + (-8) + \ldots + (-8192)$ (t_n)

$a = -2$
$r = -2$
$t_n = -8192$

$$S_n = \frac{(r t_n - a)}{(r-1)}$$

$$= \frac{[(-2)(-8192) - (-2)]}{(-2-1)}$$

$$\boxed{S_n = -5462}$$

Sum!

how many matches? $512 \div 2 = 256$

Example 3: A two-player scrabble tournament with 512 players is held. When a player loses, he/she is eliminated. The winners continue to play until a final match determines the champion. What is the total number of matches that will be played in the tournament?

$$\underbrace{256}_{\text{Round 1}} + \underbrace{128}_{\text{Round 2}} + \underbrace{64}_{\text{R.3}} + \dots + \underbrace{1}_{\text{Last Round}}$$

$\times \frac{1}{2}$ $\times \frac{1}{2}$

$a = 256$

$r = 0.5$

$t_n = 1$

$$S_n = \frac{(rt_n - a)}{(r-1)}$$

$$= \frac{((0.5)(1) - 256)}{(0.5 - 1)}$$

$$\boxed{S_n = 511}$$

511 matches will be played in total.

$100\% + 8\% = 108\%$ sum

Example 4: If a person received an 8% salary increase each year and earned a six-year total of S_n $146 718.58 by the end of the 6th year, determine the starting salary.

n n a

$a = ?$

$S_n = 146718.58$

$r = 1.08$

$n = 6$

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

$$146718.58 = \frac{a(1.08^6 - 1)}{(1.08 - 1)}$$

$$\frac{146\,718.58 \times (1.08 - 1)}{(1.08^6 - 1)} = a\cancel{(1.08^6 - 1)}$$

$$19\,999.9999 = a$$

$$20\,000 = a$$

The starting Salary was $20 000.00.

★ Units

★ Money: round to nearest 2 decimal places

★ Round at end.

Assignment p. 28 #1 (omit h), 7-8, 12-14, 16, 19

Sequences and Series Day 3: Infinite Geometric Series

An **infinite geometric series** is a geometric series that… continues forever…
∴ it does not have a final term

#1) Consider the infinite geometric series $\mathbf{1 + 2 + 4 + 8 + 16 + \dots}$ What would the sum be?

$\times 2$ $\times 2$

What is the **r** value?

$\boxed{r = 2}$

$S_1 = 1$
$S_2 = 3$
$S_3 = 7$
$S_4 = 15$
$S_5 = 31$
$S_6 = 63$
$S_7 = 127$

Sum gets infinitely large! $S_\infty = \infty$

"divergent"

#2) Consider the infinite geometric series that has $a = 4$ and $\boxed{r = \frac{1}{2}}$. Write the series up to 13 terms and find the sum for S_5, S_7, S_9, S_{11}, & S_{13}.

$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + 0.0039 \ldots$

$\times \frac{1}{2}$

$S_1 = 4$
$S_2 = 6$
$S_3 = 7$
$S_4 = 7.5$

$S_5 = 7.75$
$S_6 = 7.875$
$S_7 = 7.9375$
$S_8 = 7.96875$

$S_9 = 7.984375$
$S_{10} = 7.9921875$

The sum seems to approach 8

Convergent

When the sum approaches a fixed value, the series is said to be Convergent. When this is the case, r must be between -1 and 1.

$|r| < 1$ "little "r"" e.g. $\frac{1}{2}$, -0.4, …

vs

Divergent

If, in an infinite series, each term continues to grow, the sum does not approach a fixed value. It actually approaches infinity or negative infinity. In these situations, r is less than -1 or greater than 1. The infinite series is said to be Divergent.

$|r| > 1$ "big r" e.g. 2, 3, 4, −2, −3, 1.2, −1.5

For infinite series that are convergent, the formula for finding the sum that the series converges to is $S_\infty = \dfrac{a}{(1-r)}$

where a is the first term, r is the common ratio, and S_∞ is the sum of an infinite number of terms.

$$S_\infty = \frac{a}{(1-r)}$$

Use the formula to find the sum of the infinite series from #2 above: $4+2+1+\frac{1}{2}+\dots$

$a=4$

$r=\frac{1}{2}$

$$S_\infty = \frac{4}{\left(1-\frac{1}{2}\right)} = \frac{4}{1/2} = 4\times\frac{2}{1} = 8 \checkmark$$

Example 1: Determine whether each infinite geometric series converges or diverges. Calculate the sum.

a) $1+\frac{1}{5}+\frac{1}{25}+\cdots$ $\quad \times\frac{1}{5}$

b) $4-8+16-32+\cdots$ $\quad \times(-2)$

a) $a=1$

$r=1/5$

$$S_\infty = \frac{1}{(1-1/5)} = \frac{1}{4/5} = \frac{5}{4} \text{ or } 1.25 \text{ (convergent)}$$

$4+(-8)+16+(-32)$

b) $a=4$

$r=-2$

Divergent

Example 2: Express the repeating decimal $2.\overline{37}$ as a rational number in the form $\frac{m}{n}$.

$$2.\overline{37} = 2.\underline{37}\,\underline{37}\,37\,37\dots$$

$$= 2 + \underline{0.37} + 0.0037 + \cdots$$

$$= 2 + \frac{37}{100} + \frac{37}{10000} + \cdots \Big\}$$

Pattern

∞ geometric series!

$a=\frac{37}{100}$

$r=\frac{1}{100}$

$$= 2 + \left(\frac{37/100}{1-1/100}\right)$$

$$= 2 + \left(\frac{37/100}{99/100}\right)$$

$$= 2 + \frac{37}{\cancel{100}} \times \frac{\cancel{100}}{99}$$

$$= 2 + \frac{37}{99} \rightarrow \frac{198}{99} + \frac{37}{99} = \frac{235}{99}$$

or $= 2\frac{37}{99}$

Example 3: The first term of an infinite geometric series is 12, and the sum is 48. Determine ***r***.

$S_\infty = 48$

$a = 12$

$r = ???$

$$S_\infty = \frac{a}{(1-r)}$$

$$(1-r)\ 48 = \frac{12}{(1-r)}\ (1-r)$$

$$48(1-r) = 12$$

$$48 - 48r = 12$$

$$-48 \qquad\qquad -48$$

$$\frac{-48r}{-48} = \frac{-36}{-48}$$

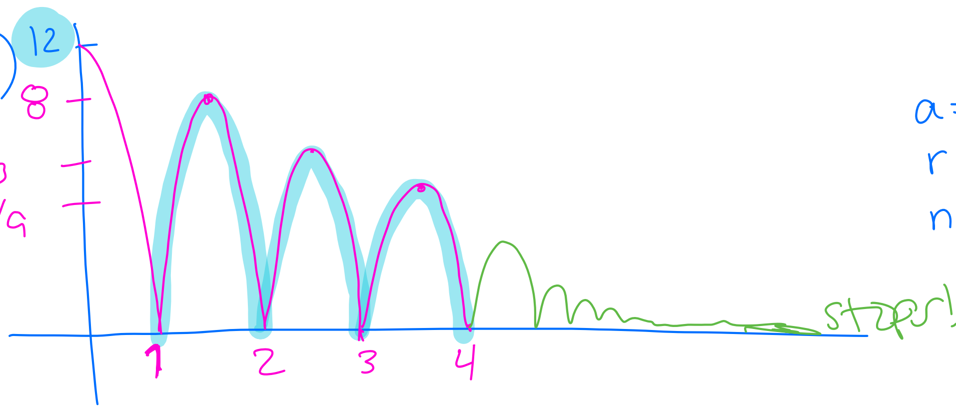
$$\boxed{r = \frac{3}{4}}$$

$r = \frac{2}{3}$

Example 4: A ball is dropped from 12 ft and rebounds $\boxed{\text{two-thirds}}$ the distance from which it fell. Find (a) the total vertical distance travelled by the ball upon hitting the floor the fourth time and (b) the total distance the ball travelled before coming to a rest.

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

a)



$a = 8$
$r = 2/3$
$n = 3$

$12 \times \frac{2}{3} = 8 \qquad 8 \times \frac{2}{3} = \frac{16}{3} \qquad \frac{16}{3} \times \frac{2}{3} = \frac{32}{9}$

$$\text{Sum} = 12 + 2(S_n)$$

$$= 12 + 2\left(\frac{8\left(\left(\frac{2}{3}\right)^3 - 1\right)}{\left(\frac{2}{3} - 1\right)}\right)$$

$$= 12 + 2(16.\bar{8})$$

$$\boxed{\text{Total dist} = 45.\bar{7} \text{ ft}}$$

$a = 8$
$r = 2/3$

b) $S_T = 12 + 2(S_\infty)$

$$= 12 + 2\left(\frac{8}{(1 - 2/3)}\right)$$

$$= 12 + 2\left(\frac{8}{1/3}\right)$$

$$= 12 + 2(24)$$

$$\boxed{\text{Total} = 60 \text{ ft}}$$

Assignment p. 35#1, 2, 4, 12, 15, 16, 20

Σ = "sigma"

Sequences and Series Day 4: Sigma Notation

Often in math we require ~~that~~ the ___Sum___ of many variables. Summation or ___Sigma___ notation is a convenient and simple form of shorthand used to give a concise expression for a sum of the values of a variable.

$\times 2$

$t_1 \; t_2 \; t_3 \; t_4 \; t_5 \; t_6$

For example, the geometric sequence with 6 terms 1+2+4+8+16+32 has a general term:

$a = 1 \quad r = 2 \quad n = 6$

$$t_n = ar^{n-1} \qquad t_n = 1(2)^{n-1} \qquad \boxed{t_k = 2^{k-1}}$$ general term

Each term can be represented as $t_1 = 1(2)^{1-1}$ $\quad t_2 = 1(2)^{2-1}$ $\quad t_3 = 1(2)^{3-1}$

$$t_4 = 1(2)^{4-1} \qquad t_5 = 1(2)^{5-1} \qquad t_6 = 1(2)^{6-1}$$

The series is the sum of all these terms. This is abbreviated to:

"Let k "run" from 1 to 6" (count 1, 2, 3, 4, 5, 6)

Σ "sum"

$$\sum_{k=1}^{6} 2^{k-1}$$ general term

The symbol $\sum$ is the Greek letter sigma. When $\sum$ is shown, it is called sigma notation.

general form for any term

Example 1: Write the series corresponding to $\sum_{k=1}^{3}(k^2 + 2k + 5)$ $= \underset{t_1}{8} + \underset{t_2}{13} + \underset{t_3}{20} = 41$

$k=1$ $\quad t_1 = (1)^2 + 2(1) + 5 = 8$

$k=2$ $\quad t_2 = (2)^2 + 2(2) + 5 = 13$

$k=3$ $\quad t_3 = (3)^2 + 2(3) + 5 = 20$

Example 2: Find the sum of the geometric series $\sum_{k=1}^{10} 3(-2)^{k-1}$ $= 3 + (-6) + 12 + \ldots$

ar^{n-1}

$k=1$ $\quad t_1 = 3(-2)^{1-1} = 3$

$k=2$ $\quad t_2 = 3(-2)^{2-1} = -6$

$k=3$ $\quad t_3 = 3(-2)^{3-1} = 12$

$a = 3$

$r = -2$

$n = 10$

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

$$= \frac{3\left[(-2)^{10} - 1\right]}{-2 - 1}$$

$$\boxed{S_{10} = -1023}$$

$$\sum_{k=1}^{n} t_k$$

$t_k = ar^{k-1}$ $\quad a=6$

$t_k = 6(3)^{k-1}$ $\quad r=3$

$t_1 \ t_2 \ t_3 \ t_4 \ t_5$

Example 3: Write the geometric series 6+18+54+162+486 using sigma notation with index k=1

$\times 3$

$$\boxed{\sum_{k=1}^{5} 6(3)^{k-1}}$$

Example 4: Find the number of terms in each finite series:

a) $\sum_{k=5}^{17}(2^k+4)$

$$\boxed{\sum_{k=a}^{b} \quad n = b-a+1}$$

b) $\sum_{k=0}^{100}(2^k+4)$

$n = 17-5+1$

$n = 13$

$n = 100-0+1$

$n = 101$

Example 5: Find the sum of the infinite series: $\boxed{S_\infty = \frac{a}{(1-r)}}$?

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{3}\right)^{k+1} = \left(-\frac{1}{9}\right) + \left(\frac{1}{27}\right) + \left(-\frac{1}{81}\right)$$

k=1

$t_1 = (-1)^1\left(\frac{1}{3}\right)^{(1+1)} = (-1)\left(\frac{1}{3}\right)^2 = -\frac{1}{9}$

k=2

$t_2 = (-1)^2\left(\frac{1}{3}\right)^{(2+1)} = (1)\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

k=3

$t_3 = (-1)^3\left(\frac{1}{3}\right)^{(3+1)} = (-1)\left(\frac{1}{3}\right)^4 = -\frac{1}{81}$

$a = -\frac{1}{9}$

$r = -\frac{1}{3}$

(✓) $|r|<1$

("r" is little)

$$S_\infty = \frac{\left(-\frac{1}{9}\right)}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{-1/9}{4/3}$$

$$= -\frac{1}{9} \times \frac{3}{4}$$

$$= \boxed{-\frac{1}{12}} = 0.08\overline{3}$$

Assignment p. 28 #2-4, p35 #3